On the modelling of dependence between univariate Lévy wear processes and impact on the reliability function

Sophie Mercier & Ghislain Verdier

UNIVERSITÉ DE PAU ET DES PAYS DE L'ADOUR

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Summary

The point

- Two components with *univariate* deterioration modeled by *non negative Lévy processes* (e.g.: gamma processes, inverse gaussian processes), $Y_i = (Y_i(t))_{t \geq 0}$, $i = 1, 2$:
  - $Y_i(0) = 0$ almost surely,
  - $(Y_i(t))_{t \geq 0}$ has independent increments,
  - $(Y_i(t))_{t \geq 0}$ is a right-continuous process with left-side limits (càdlàg).
- Comparison of three standard ways for modeling their *dependence*.
- *Impact* of the model for dependence on the *reliability* of a two-component series system.

Summary

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The model based on a regular copula

Assumption:
Let $\mathbf{Y}(t) = (Y_1(t), Y_2(t))$ for all $t \geq 0$.
Then

$$\mathbf{F}_{\mathbf{Y}(t_j)-\mathbf{Y}(t_{j-1})}(y_1, y_2) = \mathbf{C}\left(F_{Y_1(t_j)-Y_1(t_{j-1})}(y_1), F_{Y_2(t_j)-Y_2(t_{j-1})}(y_2)\right)$$

for all $j = 1, \cdots, d$, all $0 = t_0 < t_1 < \cdots < t_m$ and all $y_1, \cdots, y_d \in \mathbb{R}_+$,
where $\mathbf{C}$ is a *time independent* regular copula [Nelsen 2006].

Remark: under this assumption, some papers compute the likelihood function as if the bivariate increments $\mathbf{Y}(t_1)$, $\mathbf{Y}(t_2) - \mathbf{Y}(t_1), \cdots, \mathbf{Y}(t_d) - \mathbf{Y}(t_{d-1})$ were *independent*.

Question: is this coherent?

[Duan et al. 2018, Fang et Pan 2021, Fang et al. 2020, Liu et al. 2014, Palayangoda et Ng 2021, Pan et al. 2020, Wang et al. 2013, Zhou et al. 2010]

Coherence of the model based on a regular copula (1/2)

- Let $0 < t_1 < t_2$. Then:

$$\mathbf{Y}(t_2) = \mathbf{Y}(t_1) + (\mathbf{Y}(t_2) - \mathbf{Y}(t_1)).$$

- *If* the process $\mathbf{Y} = (\mathbf{Y}_t)_{t \geq 0}$ has *independent increments*, then

$$\mathbf{F}_{\mathbf{Y}(t_2)} = f_{\mathbf{Y}(t_1)} * \mathbf{F}_{\mathbf{Y}(t_2)-\mathbf{Y}(t_1)}.$$

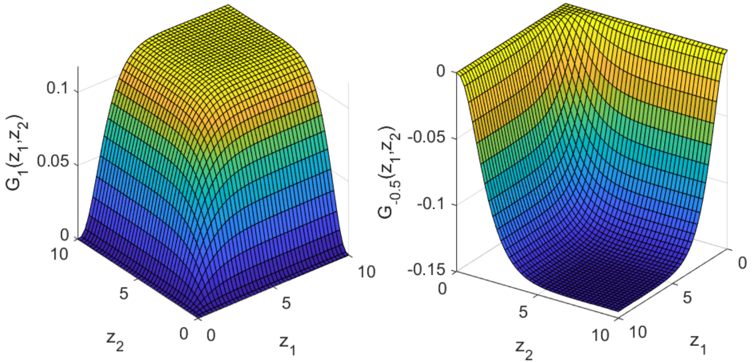
- This implies $\mathbf{G}(z_1, z_2) = 0$ with

$$\begin{aligned}\mathbf{G}(z_1, z_2) = & \int_{[0,z_1]\times[0,z_2]} \mathbf{C}\left(F_{Y_1(t_2)}(y_1), F_{Y_2(t_2)}(y_2)\right) dy_1\ dy_2 \\ & - \int_{[0,z_1]\times[0,z_2]} \mathbf{C}\left(F_{Y_1(t_2)-Y_1(t_1)}(x_1), F_{Y_2(t_2)-Y_2(t_1)}(x_2)\right) \\ & \times \mathbf{C}\left(F_{Y_1(t_1)}(z_1 - x_1), F_{Y_2(t_1)}(z_2 - x_2)\right) dx_1\ dx_2\end{aligned}$$

for all $(z_1, z_2) \in \mathbb{R}_+^2$.

Coherence of the model based on a regular copula (2/2)

- $Y_i(t) \sim \Gamma(t, 1)$ for $i = 1, 2$,
- C : Clayton copula with parameter $\theta = 1$ (left) and $\theta = -0.5$ (right),
- $t_1 = 1$, $t_2 = 2$.



Conclusion: $\mathbf{Y}$ does not have independent increments. Hence:
- What about the joint distribution of $(\mathbf{Y}(t_1), \mathbf{Y}(t_2) - \mathbf{Y}(t_1), \cdots, \mathbf{Y}(t_d) - \mathbf{Y}(t_{d-1}))$???
- The construction is uncomplete...

The model constructed through superposition

Let $(X_i(t))_{t \geq 0}$, $i = 1, 2, 3$ be independent univariate non negative Lévy processes.

We set:

$$\begin{cases} Y_1(t) = X_1(t) + X_3(t), \\ Y_2(t) = X_2(t) + X_3(t). \end{cases}$$

Then:

- $\mathbf{Y} = (Y_1, Y_2)$ is a *bivariate Lévy process*.
- the joint *pdf* and *cdf* of $\mathbf{Y}(t)$ are available *in full form*.

Remark:

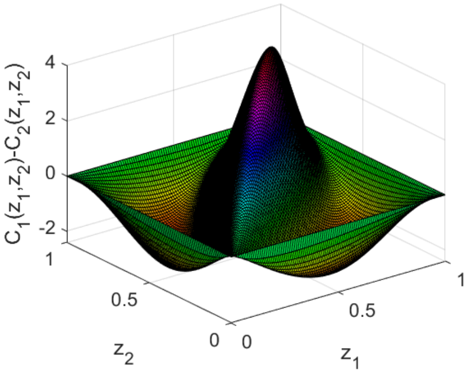
- All the dependence comes from $X_3(t)$, which leads to simultaneous and equal size jumps in the two marginal processes.
- Bivariate Lévy measure:

$$\mu(d\mathbf{y}) = \mu_{X_1}(y_1)\, dy_1 \times \delta_0(dy_2) + \delta_0(dy_1) \times \mu_{X_2}(y_2)\, dy_2 + \mu_{X_3}(y_1)\, dy_1 \times \delta_{y_1}(dy_2).$$

[Liu et al. 2021, Mercier & al. 2012]

On the time-dependence of $\mathbf{C}_t$ in the superposition model

- Let $X_i(t) \sim \Gamma(t, 1)$ for $i = 1, 2, 3$.
- Then: $Y_i(t) = X_i(t) + X_3(t) \sim \Gamma(2t, 1)$ for $i = 1, 2$.
- Let $\mathbf{C}_t$ be the regular copula that links the r.v.s $Y_1(t)$ and $Y_2(t)$.
- Plot of $\mathbf{C}_1 - \mathbf{C}_2$ (with $t = 1$ and $t = 2$):



Conclusion: even for such a simple bivariate Lévy process, the regular copula $\mathbf{C}_t$ *depends* on t.

The model based on a Lévy copula

- Let $(\mathbf{Y}(t) = (Y_1(t), Y_2(t)))$ be a bivariate non-decreasing Lévy process.
- Bivariate Lévy measure: $\boldsymbol{\mu}(d\mathbf{z})$.
- Univariate Lévy measures: $\mu_i(dz_i)$, $i = 1, 2$.
- Tail integral functions:

$$\mathbf{U}(\mathbf{y}) = \mathbf{U}(y_1, y_2) = \iint_{[y_1,+\infty)\times[y_2,+\infty)} \boldsymbol{\mu}(d\mathbf{z}),$$

$$U_i(y_i) = \int_{[y_i,+\infty)} \mu_i(dz_i) \text{ for } i = 1, 2.$$

- All the dependence between Y_1 and Y_2 is captured by a Lévy copula **L**, such that

$$\mathbf{U}(\mathbf{y}) = \mathbf{L}(U_1(y_1), U_2(y_2))$$

for all y_1, $y_2 > 0$ (Sklar's theorem for Lévy processes).

[Cont et Tankov 2004, Tankov 2016]

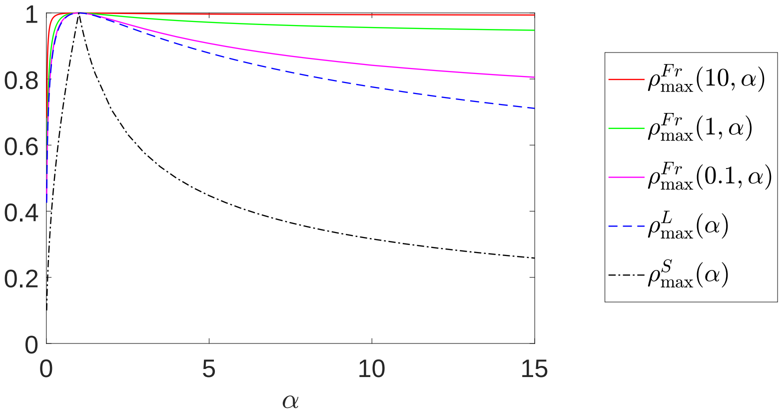
Numerical assessment of the model based on a Lévy copula

- The joint distribution of $\mathbf{Y}(t) = (Y_1(t), Y_2(t))$ is usually not avalaible in full form.
- Numerical assessment:
  - through approximate Monte-Carlo simulations (or numerical schemes).
- Comparison between two Monte-Carlo schemes:
  - [Grothe et Hofert 2015], valid for Archimedean Lévy copula,
  - [Tankov 2016], symmetric conditional method.
  - Both MC schemes perform well, with a slight advantage to [Tankov 2016], especially in case of a low dependence.

Comparison of the dependence range

Maximal correlation coefficient $\rho_{\max}$ between $Y_1(t)$ and $Y_2(t)$

- Let $Y_i(t) \sim \Gamma(A_i t, 1)$, $i = 1, 2$ and $\alpha = A_1/A_2$.
- L: Lévy copula.
- S: superposition.
- Fr: upper Fréchet bound for regular copula with $t = 0.1$, $t = 1$ and $t = 10$.



The three models: summary

- Model based on regular copulas:
  - the bivariate process does not have independent increments,
  - the joint distribution of $(\mathbf{Y}(t_1), \mathbf{Y}(t_2) - \mathbf{Y}(t_1), \cdots, \mathbf{Y}(t_d) - \mathbf{Y}(t_{d-1}))$ is not specified,
  - hence: the construction is incomplete.

- Model constructed by superposition:
  - provides a simple bivariate Lévy process,
  - dependence range reduced with respect to a general Lévy process,

- Model based on Lévy copula:
  - provides a most general Lévy process,
  - requires approximate Monte-Carlo simulations (or numerical schemes) for its numerical assessment.

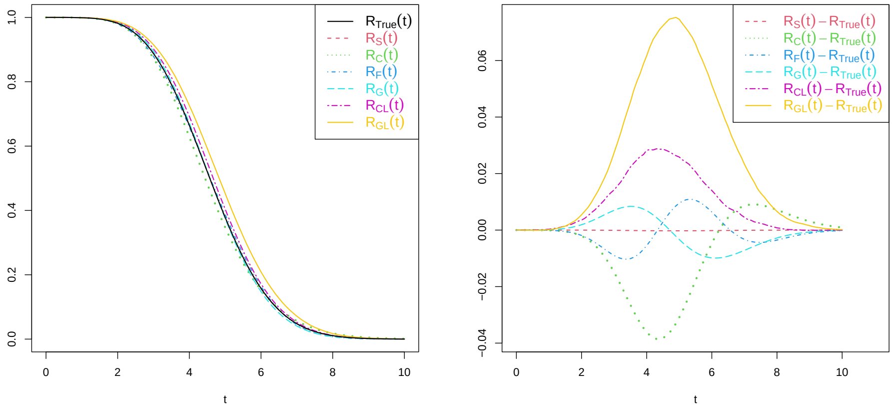
Summary

Methodology

- Two-component series system.
- Joint deterioration level of the two components modeled by one of the three models.
- Component j is failed as soon as $Y_j(t) \geq L_j$, $j = 1, 2$.
- Reliability: $R(t) = \mathbb{P}(Y_1(t) < L_1, Y_2(t) < L_2)$.
- Comparison methodology:
  - Data are generated from one of the two coherent models (superposition or Lévy copula): the "true" model.
  - Periodic ($\Delta t_i = 1$) or non periodic ($\Delta t_i \sim \mathcal{U}([1, 20])$) observation times.
  - Marginal parameters: assumed to be known.
  - Parametric Lévy / regular copula families and for the superposition model: one single dependence parameter to estimate.
  - The reliability is next computed with the estimates for the three models.

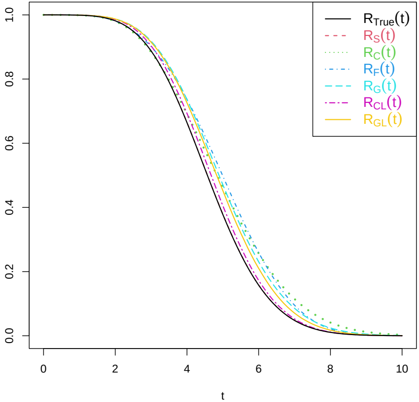
True model: superposition - Periodic observation

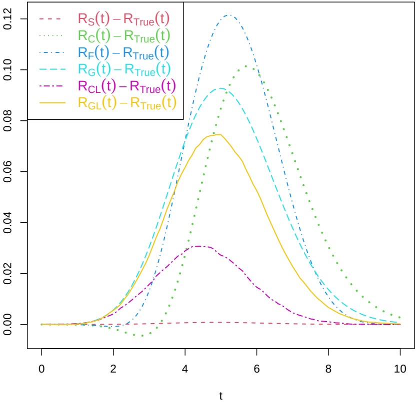
- S = superposition,
- $C/F/G$ = Clayton/Fréchet/Gumbel regular copula,
- CL/GL = Clayton/Gumbel Lévy copula.



True model: superposition - Aperiodic observation

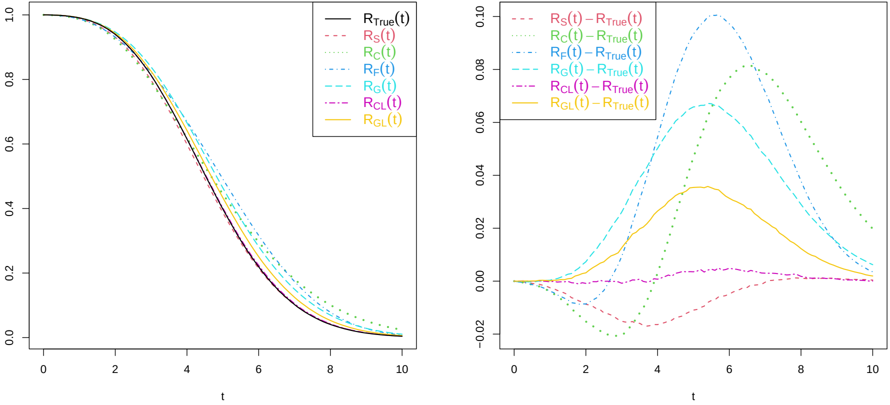
- S = superposition,
- $C/F/G$ = Clayton/Fréchet/Gumbel regular copula,
- CL/GL= Clayton/Gumbel Lévy copula.





True model: Clayton-Lévy copula - Aperiodic observation

- S = superposition,
- $C/F/G$ = Clayton/Fréchet/Gumbel regular copula,
- CL/GL= Clayton/Gumbel Lévy copula.



On the impact of a wrong modeling on the reliability function: Summary

- True model = superposition:
  - Not that bad results for the model based on regular copulas in case of *periodic* observations (expected as the observations are i.i.d.); less convincing for *aperiodic* observations.
  - A *smooth* (Archimedean) Lévy copula does not catch that well the dependence implied by the superposition model.
  - This can be due to the fact that the bivariate Lévy measure for the superposition model is not absolutely continuous with respect to Lebesgue measure.
- True model = Lévy copula:
  - A wrong choice for the Lévy copula family has a clear impact on the reliability function.
  - The superposition model can even provide better results than a wrongly chosen Lévy copula (but not the model based on regular copulas).

Summary

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How to choose the dependence model?

- No theoretical basis for the model based on regular copulas:
  - we would suggest not to use this model.
- To choose between a Lévy process constructed by superposition or a Lévy copula:
  - estimate Pearson correlation coefficient.
  - refer to technical experts to better understand the interaction between the two marginal indicators.
  - development of goodness-of-fit tests???
- In this presentation:
  - low-frequency observations.
- Whenever high-frequency observations are available:
  - consider non parametric estimation for the underlying Lévy copula.

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