Degradation Model Selection Using Depth Function

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20.05.2022

Degradation Modeling

- Reliability
- Advanced Industries
- Lifetime Prediction
- Degradation Data
- Degradation Modeling
- The Best Model
- Statistical Criteria
- Depth Function

Degradation Modeling

- Many ***failure*** mechanisms in engineering, medical, social, and economic settings can be traced to an underlying ***degradation*** process.
- In an engineering setting, degradation is the irreversible accumulation of damage throughout life that leads to failure.

Degradation Modeling

- The stochastic-process-based models show great flexibility in describing the failure mechanisms.
- A ***stochastic process*** $\boldsymbol{X} = \{X(t), t \in T\}$ on $(\Omega, \mathcal{T}, \mathbb{P})$ is a collection of Random Variables.
- The observation of a stochastic process at time t is denoted by $\boldsymbol{x}(t)$
- The index set

$$T = \begin{cases} \text{countable set} & X(t) : \text{a discrete-time stochastic process} \\ \text{non-countable set} & X(t) : \text{a continuous-time stochastic process} \end{cases} \quad (1)$$

- $t \in T$ is considered as ***time***

Lévy Process

(1) It has right continuous sample paths with left limits.

(2) It has stationary increments. That is, for any $t, s \geq 0$, $X(t+s) - X(t)$ does not depend on t.

(3) It has independent increments. That is, for every pair of disjoint intervals $(t_1, t_2), (t_3, t_4)$, with $t_1 < t_2 < t_3 < t_4$, the random variables $X(t_2) - X(t_1)$ and $X(t_4) - X(t_3)$ are independent.

(4) $X(0) = 0$ with probabilty 1

Wiener Process

A Wiener process $\{X(t), t \geq 0\}$ is a Lévy process with

$$X(t+u) - X(t) \sim \mathcal{N}(0, u) \tag{2}$$

A wiener process with linear drift is a stochastic differential equation:

$$dX(t) = \mu dt + \sigma dB(t) \tag{3}$$

where $B(t)$ is a Standard Brownian Motion and $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$.

Gamma Process

A Gamma Process $\{X(t), t \geq 0\}$ is a Lévy process with

$$X(s) - X(t) \sim Gamma(v(s) - v(t), u) \quad \forall s > t \geq 0 \qquad (4)$$

where $X \sim Gamma(k, u)$ with PDF

$$f(x|u,k) = \frac{1}{\Gamma(k)u^k} x^{k-1} e^{-\frac{x}{u}}, \quad x > 0, k > 0, u > 0 \qquad (5)$$

and $v(t)$ is a non-decreasing, right-continuous, real-valued function.

Inverse Gaussian Process

An Inverse Gaussian process $\{X(t), t \geq 0\}$ is a Lévy process with

$$X(s) - X(t) \sim IG(\Lambda(s) - \Lambda(t), \eta(\Lambda(s) - \Lambda(t))^2) \quad s > t \geq 0 \quad (6)$$

where $\Lambda(t)$ is a monotone increasing function and for $X \sim IG(a, b), a, b > 0$ the PDF is

$$f(x|a,b) = \sqrt{\frac{b}{2\pi x^3}} \exp\left[-\frac{b(x-a)^2}{2a^2 x}\right] \quad x > 0 \quad (7)$$

Stochastic Differential Equations (SDE)

- A differential equation with stochastic process terms.

The process $\{X(t), t \geq 0\}$ is called a ***diffusion process*** if

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dB(t) \tag{8}$$

is satisfied, where
$B(t)$: the Standard Brownian Motion,
$\mu(X(t), t)$: the ***drift*** function,
$\sigma(X(t), t)$: the ***diffusion*** function.

Bessel Process

The Bessel process satisfies the following SDE:

$$dX(t) = \frac{a}{X(t)}dt + dB(t), \tag{9}$$

where $a \in \mathbb{R}$. Also,

$$X(t) = \|\boldsymbol{W(t)}\|_2 = \sqrt{\sum_{i=1}^{d}(W_i(t))^2}. \tag{10}$$

where $a = (d-1)/2$ and $\boldsymbol{W(t)} = (W_1(t), W_2(t), \ldots, W_d(t))$ is a d-dimensional Wiener process and $\|.\|_2$ stands for l^2-norm.

Assumptions

- Let $(\Omega, \mathcal{T}, \mathbb{P})$ be a probability space.
- Let $\boldsymbol{X} = \{X(t), t \in T\}$ be on $(\Omega, \mathcal{T}, \mathbb{P})$.
- Let $\mathcal{M}(T)$ be the set of real valued functions on T.
- Let T_0 be a compact subset of T.
- Let P be the distribution of the $\boldsymbol{X} = \{X(t), t \in T\}$.

Depth Function

- Depth function was originally introduced to extend the notion of **median** to multi-variate random variables.
- The notion of depth has been extended to **Functional Data**.
- Stochastic processes paths can be considered as functional data.

Depth Function

Definition

For a function $h \in \mathcal{M}(T_0)$ the Depth Function with respect to P is defined as

$$\begin{aligned} D_P : \mathcal{M}(T_0) &\to [0,1] \\ h &\to D_P(h) \end{aligned}$$

The **closer** the quantity $D_P(h)$ to **1**, the **deeper** the observation h is recognized with respect to the distribution P.

Convex hull

Definition

The **convex hull** of a set of points S in $\mathbb{R}^n$ is the intersection of all convex sets containing S. For N points $p_1, \ldots, p_N$, the convex hull is then given by the expression:

$$\left\{ \sum_{j=1}^{N} \lambda_j p_j, \ \sum_{j=1}^{N} \lambda_j = 1, \ \ \lambda_j > 0, \forall j \right\}$$

Graph

Definition

The **graph** of any function $f \in \mathcal{M}(T_0)$ is defined by

$$graph(f) = \{(t, y) : y = f(t), t \in T_0\},$$

and

$$graph\{(f_1, f_2, \ldots, f_n)\} = \cup_{i=1}^{n} graph(\{f_i\}).$$

ACH Depth

Definition

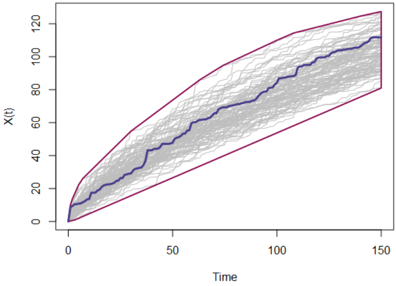
Suppose that $\boldsymbol{X_1}, \boldsymbol{X_2}, \ldots$ are independent and identically distributed copies of $\boldsymbol{X}$. Let $J \geq 1$ be a fixed integer, the **ACH** (The Area of the Conves Hull) depth of degree J with respect to P is defined by

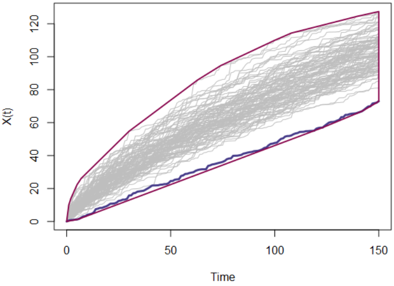
$$D_{J,P} : \mathcal{M}(T_0) \rightarrow [0,1]$$

$$h \rightarrow \mathbb{E}\left[\frac{\lambda(conv(graph(\{\boldsymbol{X_1}, \ldots, \boldsymbol{X_J}\})))}{\lambda(conv(graph(\{\boldsymbol{X_1}, \ldots, \boldsymbol{X_J}\} \cup \{h\})))}\right]$$

where $conv(A)$ denotes the convex hull of any subset $A \in \mathcal{M}(T_0)$ and λ denotes the Lebesgue measure on $\mathbb{R}^2$.

ACH Depth


X(t)
0
20
40
60
80
100
120
0
50
100
150
Time


X(t)
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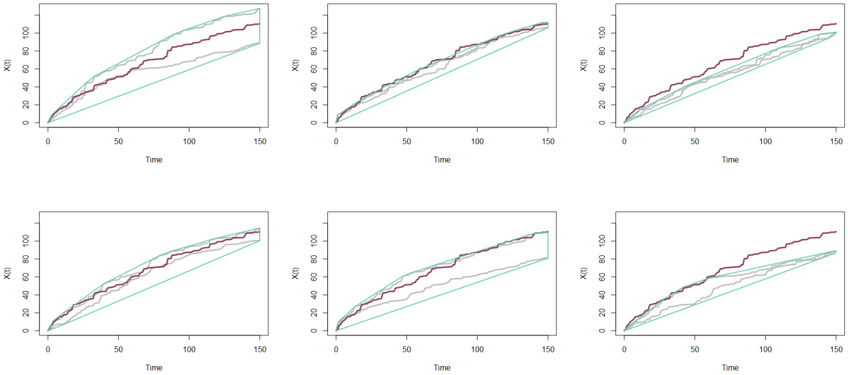
Time

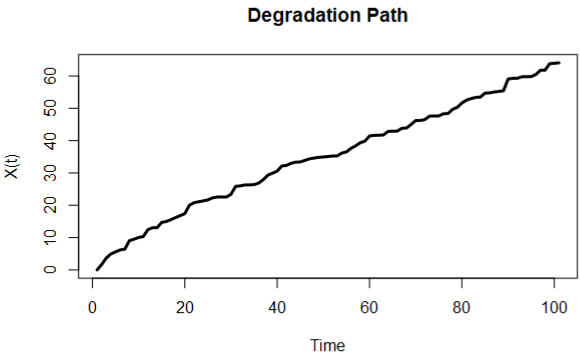
ACH Depth



Simulation Study

(1) Degradation path observation
(2) Proposing degradation models
(3) Calculating the depth function for each model
(4) Analyzing each model based on the First Hitting Times (FHT)

Observing the degradation path



Observing a degradation path from the Gamma Model

Proposing degradation models

- Gamma
- Inverse Gaussian
- Bessel

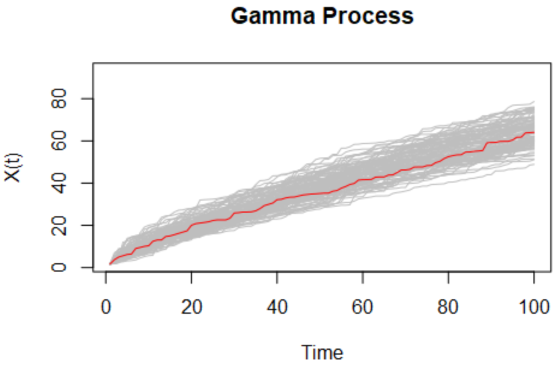
Depth function

Does the depth function assigns the maximum value to the **Gamma** model?

Depth function

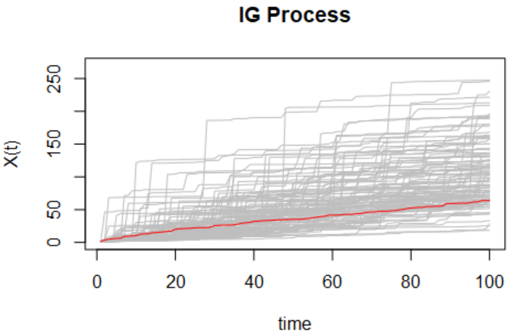
- The observation comes from a known distribution with unknown parameters.
- The parameter are estimated by **Maximum Likelihood** method for the proposed models.
- Random sample paths from each model with estimated parameters are generated.
- The depth function value of the observed path with respect to each model is calculated.

Depth function



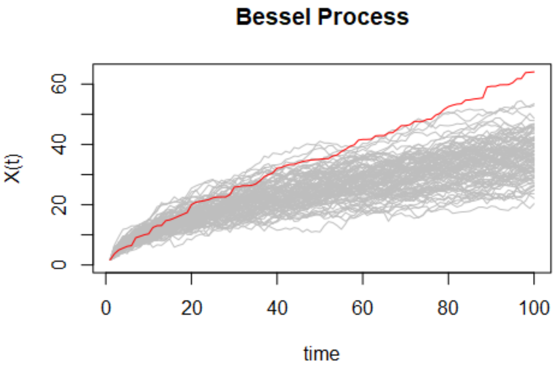
Sample paths of the Gamma model (gray)
Observed sample path (red)

Depth function



Sample paths of the Inverse Gaussian model (gray)
Observed sample path (red)

Depth function



Sample paths of the Bessel model (gray)
Observed sample path (red)

Depth function

Table: Depth Function Values

	ACH
Gamma	0.9343
IG	0.9174
Bessel	0.6154

The **gamma** model has the highest value of the depth function and the **IG** model is in the second place with a small difference, but for the **Bessel** model the value of the depth function is less than the other two models.

Model Analysis

The quantiles of the First Hitting Times of the observed level $L = 5.57 = \boldsymbol{x}(5)$

Quantiles	25%	50%	75%	100%
Gamma	4	5	6	13
IG	7	12	17	45
Bessel	3	4	4	11

The Mean, Mode and Variance of the First Hitting Times of the observed level $L = 5.57 = \boldsymbol{x}(5)$

	Mean	Mode	Var
Gamma	5.02	4	2.78
IG	12.63	6	57.29
Bessel	3.85	3	1.25

Gamma: the median mean and mode of the FHTs are slightly different from the FHT of the observed path.
IG: the quantiles, mean and mode of the FHTs are larger than the FHT of the observed path.
Bessel: the quntiles, mean and mode of FHTs show a little difference with the FHT of the observed path but they are not as good as the gamma model.

Model Analysis

The quantiles of the First Hitting Times of the observed level $L = 10.02 = \boldsymbol{x}(10)$

Quantiles	25%	50%	75%	100%
Gamma	8	9	11	23
IG	10	18	27	68
Bessel	7	9	11	41

The Mean, Mode and Variance of the First Hitting Times of the observed level $L = 10.02 = \boldsymbol{x}(10)$

	Mean	Mode	Var
Gamma	9.72	9	8.54
IG	19.31	14	135
Bessel	9.47	8	9.9

The quantiles of the **IG** model show that only 25% of the FHTs are less than 10, and if L = 10.02 is the failure level, this model predicts the failure time very late. The median, mean, and mode of FHTs of the **gamma** and **bessel** models are moderately different from the FHT of the observed path but the **Bessel** model has the larger variance.

Model Analysis

The quantiles of the First Hitting Times of the observed level $L = 34.92 = \boldsymbol{x}(50)$

Quantiles	25%	50%	75%	100%
Gamma	40	46	52	86
IG	28	45	62	100
Bessel	19	24	30	88

The Mean, Mode and Variance of the First Hitting Times of the observed level $L = 34.92 = \boldsymbol{x}(50)$

	Mean	Mode	Var
Gamma	46.26	46	74.21
IG	45.75	48	543.91
Bessel	45.57	35	293.82

The median, mean and mode of FHTs for the **gamma** and **IG** are close to each other, but for the **IG** the variance of the FHTs at this level is much larger than the **gamma**. Values for the **Bessel** model indicate that this model cannot well explain the observation path. Because the more time passes, the larger the difference between the FHTs of the **Bessel** model and the FHT of the observed sample path.

The Future of the Research

- Analyzing the ACH depth function with different values of J
- Analyzing the depth function applying on a sample path with a few number of points
- Analyzing the depth function applying on a sample path with nonequidistant time intervals
- Studying other depth functions
- Formulate a weighted model based on the depth function

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Thanks For Your Attention