Minimal sample size in balanced ANOVA models and its calculation using the R package “miniSize”

Bernhard Spangl

Institute of Statistics
Department of Landscape, Spatial and Infrastructure Sciences
University for Natural Resources and Life Sciences Vienna

ENBIS-22, Trondheim, June 28, 2022

Contents

3 Motivating Examples

- **Nested classification** (Symbol $A \succ B \succ C$): The levels of C are nested within B, and those of B are nested within A.
  Example (Rasch et al., 2011): the levels of the factor A be farms in a certain country with herds of milk cattle with the same number of bulls and the same number of cows per bull inseminated by this bull. The bulls are the levels of the factor B nested within farms and the cows are the levels of the factor C nested within bulls and farms. The milk performance of a certain number of daughters of the cows is measured.

3 Motivating Examples (cont.)

- **Mixed classification 1** (Symbol $(A \times B) \succ C$): A and B are cross-classified, and C is nested within the classes (i, j).
  Example (Wille et al., 2002): The levels of factor A are different lipid contents of feeds in a fish farming study. The factor B models the treatment with growth hormone (GH) at different levels. These two factors are crossed. The factor C is the tank effect nested within the lipid diet and GH treatment interaction. Several growth and feeding factors were measured.

3 Motivating Examples (cont.)

- **Mixed classification 2** (Symbol $(A \succ B) \times C$): B is nested in A, and C is cross-classified with all $A \succ B$ combinations.
  Example (Underwood, 1993): We want to detect environmental impacts that occur as a result of a planed disturbance. The levels of factor A are before and after the disturbance. One or more ecological variables, such as abundances of a popluation, chemical concentrations, rate of production of detoxifying enzymes or precentage of pathological cells, were measured at different sampling times (levels of factor B) and different locations (levels of factor C). The sampling is done before and after the disturbance and thus B is nested within factor A. As all locations (the disturbed one as well as several control locations) were examined before and after the disturbance, the factor C is crossed with sampling times nested within before or after the disturbance. The factors sampling time and location may be considered random.

Main Results

We consider balanced one-, two- or three-way ANOVA models with at least one fixed factor A (cf. Spangl et al., 2021).

- We derive the details for the noncentrality parameter (Result 1).
- We derive the worst case noncentrality parameter, required to obtain the guaranteed power of an ANOVA experiment (Result 2).
- We show how to determine the minimal experimental size for ANOVA experiments by a new structural result that we call "pivot" effect (Result 3). The "pivot" effect means one of the parameters (the "pivot" parameter) is more power-effective than the others. In fact, the "pivot" effect shows that for some models not the number of replicates n but a different parameter should be increased to achieve any given prespecified power.

Example of Result 1

For the model $A \succ \boldsymbol{B} \succ \boldsymbol{C}$, Result 1 states that the test statistic $\boldsymbol{F}_A = \boldsymbol{MS}_A / \boldsymbol{MS}_{B \text{ in } A}$ has an F-distribution (central under H_0, noncentral in general) with numerator d.f. $df_1 = a - 1$, denominator d.f. $df_2 = a(b-1)$, and noncentrality parameter

$$\lambda = RS/T = b \cdot \frac{\sum_i \alpha_i^2}{\sigma^2_{\beta(\alpha)} + \frac{1}{c}\sigma^2_{\gamma(\alpha\beta)} + \frac{1}{cn}\sigma^2} .$$

Complete List (Result 1)

Table 1: List of 1-, 2- and 3-way ANOVA models with fixed factor A, for use in Theorem 2.1 etc. The letters $a, b, \dots$ denote the numbers of levels, and n is the number of replicates. To point out equivalences, the variance component notation is simplified, such as $\sigma^2_{\alpha,\beta}$ represents both $\sigma^2_{\alpha\beta}$ and $\sigma^2_{\beta(\alpha)}$. In the first column, bold font indicates random factors. The "pivot" parameter, also printed in bold to indicate randomness, is the most power-effective parameter, see Theorem 2.7.

Model	Pivot parameter	df_1	df_2	$\lambda = RS/T$		
				R	S	T
A	$\boldsymbol{n}$	$a-1$	$a(n-1)$	n	$\sum_i \alpha_i^2$	σ^2
$A \times B$	$\boldsymbol{n}$	“	$ab(n-1)$	bn	“	“
$A \succ B$	“	“	“	“	“	“
$A \times \boldsymbol{B}$	$\boldsymbol{b}$	“	$(a-1)(b-1)$	b	“	$\sigma^2_{\alpha,\beta} + \frac{1}{n}\sigma^2$
$A \succ \boldsymbol{B}$	“	“	$a(b-1)$	“	“	“
$V \succ A$	$\boldsymbol{n}$	$v(a-1)$	$va(n-1)$	n	$\sum_{i,j} \alpha_{i(j)}^2$	σ^2
$\boldsymbol{V} \succ A$	“	“	“	“	“	“
$A \times B \times C$	$\boldsymbol{n}$	$a-1$	$abc(n-1)$	bcn	$\sum_i \alpha_i^2$	σ^2
$A \succ B \succ C$	“	“	“	“	“	“
$(A \times B) \succ C$	“	“	“	“	“	“
$(A \succ B) \times C$	“	“	“	“	“	“
$A \times (B \succ C)$	“	“	“	“	“	“
$A \succ B \succ \boldsymbol{C}$	$\boldsymbol{c}$	“	$ab(c-1)$	bc	“	$\sigma^2_{\alpha,\beta,\gamma} + \frac{1}{n}\sigma^2$
$(A \times B) \succ \boldsymbol{C}$	“	“	“	“	“	“
$A \times (B \succ \boldsymbol{C})$	“	“	$(a-1)b(c-1)$	“	“	“
$(A \succ B) \times \boldsymbol{C}$	“	“	$(a-1)(c-1)$	c	“	$\sigma^2_{\alpha,\gamma} + \frac{1}{bn}\sigma^2$
$A \times \boldsymbol{B} \times C$	$\boldsymbol{b}$	“	$(a-1)(b-1)$	b	“	$\sigma^2_{\alpha,\beta} + \frac{1}{cn}\sigma^2$
$(A \times \boldsymbol{B}) \succ C$	“	“	“	“	“	“
$A \times (\boldsymbol{B} \succ C)$	“	“	“	“	“	“
$A \succ \boldsymbol{B} \succ C$	“	“	$a(b-1)$	“	“	“
$(A \succ \boldsymbol{B}) \times C$	“	“	“	“	“	“
$A \succ \boldsymbol{B} \succ \boldsymbol{C}$	$\boldsymbol{b}$	“	“	“	“	$\sigma^2_{\alpha,\beta} + \frac{1}{c}\sigma^2_{\alpha,\beta,\gamma} + \frac{1}{cn}\sigma^2$
$(A \times \boldsymbol{B}) \succ \boldsymbol{C}$	“	“	$(a-1)(b-1)$	“	“	“
$A \times (\boldsymbol{B} \succ \boldsymbol{C})$	“	“	“	“	“	“
$V \succ A \succ B$	$\boldsymbol{n}$	$v(a-1)$	$vab(n-1)$	bn	$\sum_{i,j} \alpha_{i(j)}^2$	σ^2
$(V \succ A) \times B$	“	“	“	“	“	“
$\boldsymbol{V} \succ A \succ B$	“	“	“	“	“	“
$(\boldsymbol{V} \succ A) \times B$	“	“	“	“	“	“
$V \succ A \succ \boldsymbol{B}$	$\boldsymbol{b}$	“	$va(b-1)$	b	“	$\sigma^2_{\nu,\alpha,\beta} + \frac{1}{n}\sigma^2$
$\boldsymbol{V} \succ A \succ \boldsymbol{B}$	“	“	“	“	“	“
$(V \succ A) \times \boldsymbol{B}$	“	“	$v(a-1)(b-1)$	“	“	“
$(\boldsymbol{V} \succ A) \times \boldsymbol{B}$	“	“	“	“	“	“
$U \succ V \succ A$	$\boldsymbol{n}$	$uv(a-1)$	$uva(n-1)$	n	$\sum_{i,j,k} \alpha_{i(jk)}^2$	σ^2
$(U \times V) \succ A$	“	“	“	“	“	“
$\boldsymbol{U} \succ V \succ A$	“	“	“	“	“	“
$U \succ \boldsymbol{V} \succ A$	“	“	“	“	“	“
$(U \times \boldsymbol{V}) \succ A$	“	“	“	“	“	“
$\boldsymbol{U} \succ \boldsymbol{V} \succ A$	“	“	“	“	“	“
$(\boldsymbol{U} \times \boldsymbol{V}) \succ A$	“	“	“	“	“	“

3

Least Favorable Case (Result 2)

For an exact F-test, the computation of the power is immediate: given the type I risk α, obtain the type II risk β by solving

$$F_{df_1,df_2;1-\alpha} = F^{\lambda}_{df_1,df_2;\beta} ,$$

where $F^{\lambda}_{\nu_1,\nu_2;\gamma}$ denotes the γ-quantile of the F-distribution with degrees of freedom ν_1 and ν_2 and noncentrality parameter λ.

Result 2 We have the following lower bound for the noncentrality parameter λ. With the parameter or product of parameters denoted R of Result 1 we have

$$\lambda \geq \frac{R}{2} \cdot \frac{\delta^2}{\sigma_y^2}.$$

δ denote the minimum difference to be detected between the smallest and the largest treatment effects, i.e.,

$$\delta = \alpha_{\text{max}} - \alpha_{\text{min}} .$$

Example of Result 2

For the model $A \succ \boldsymbol{B} \succ \boldsymbol{C}$, from Result 1 we have

$$T = \sigma^2_{\beta(\alpha)} + \frac{1}{c}\sigma^2_{\gamma(\alpha\beta)} + \frac{1}{cn}\sigma^2.$$

All variance components occur in T, thus

$$\sigma_y^2 = \sigma^2_{\beta(\alpha)} + \sigma^2_{\gamma(\alpha\beta)} + \sigma^2.$$

Since $R = b$, by Result 2 we obtain for the noncentrality parameter λ,

$$\lambda \geq \frac{b}{2} \cdot \frac{\delta^2}{\sigma_y^2} \ .$$

Minimal Sample Size (Result 3)

Result 3 For given power requirements P, the minimal sample size can be obtained by varying only one parameter, which we call “pivot” parameter, keeping the other parameters minimal.

(i) If a parameter increases, then the power increases most if it is the “pivot” parameter.

(ii) For fixed size, then the maximal power occurs if the “pivot” parameter varies and the other parameters are minimal.

(iii) For fixed power, then the minimum size occurs if the “pivot” parameter varies and the other parameters are minimal.

Example of Result 3

The number of levels of the fixed factor A is $a = 6$, the minimum difference to be detected between the smallest and the largest treatment effects is $\delta = 1$, and $\alpha = 0.05$. The variance components are $(\sigma^2_{\beta(\alpha)}, \sigma^2_{\gamma(\alpha\beta)}, \sigma^2) = (1/18, 1/9, 1/6)$.

- Model $A \succ \boldsymbol{B} \succ \boldsymbol{C}$, pivot $\boldsymbol{b}$

$(\boldsymbol{b}, c, n)$	df_1	df_2	λ	P
$(\mathbf{2}, 2, 6)$	5	6	8.	0.271516
$(\mathbf{2}, 3, 4)$	5	6	9.3913	0.314513
$(\mathbf{2}, 4, 3)$	5	6	10.2857	0.342042
$(\mathbf{2}, 6, 2)$	5	6	11.3684	0.375051
$(\mathbf{3}, 2, 4)$	5	12	11.3684	0.527472
$(\mathbf{3}, 4, 2)$	5	12	14.4	0.642402
$(\mathbf{4}, 2, 3)$	5	18	14.4	0.712478
$(\mathbf{4}, 3, 2)$	5	18	16.6154	0.781856
$(\mathbf{6}, 2, 2)$	5	30	19.6364	0.897849

$P_{\text{requ.}}$	$(\boldsymbol{b}, c, n)$	df_1	df_2	λ	P
0.8	$(\mathbf{5}, 2, 2)$	5	24	16.3636	0.808263
0.85	$(\mathbf{6}, 2, 2)$	5	30	19.6364	0.897849
0.9	$(\mathbf{7}, 2, 2)$	5	36	22.9091	0.948655
0.95	$(\mathbf{8}, 2, 2)$	5	42	26.1818	0.97543

The Approximate Case

For the model

$$(A \succ \boldsymbol{B}) \times \boldsymbol{C}, \tag{1}$$

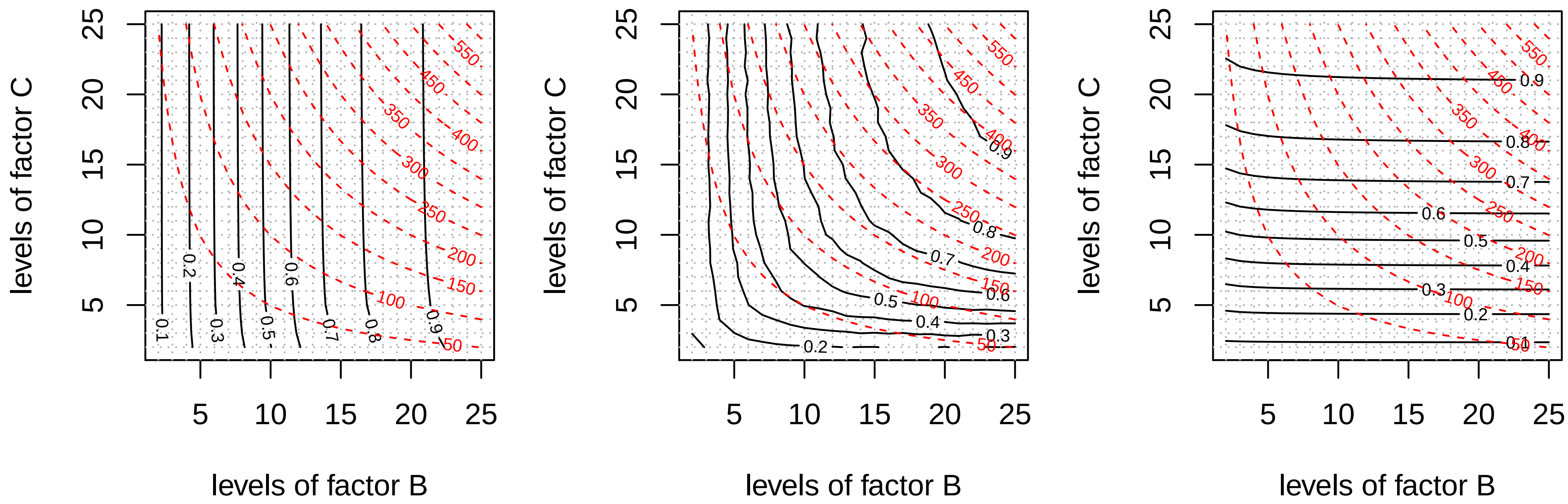
no exact F-test exists. Approximate F-tests can be obtained by Satterthwaite's approximation (Satterthwaite, 1946).

The approximate F-test d.f. involve mean squares to be simulated. We may approximate the power of the test by simulating data such that H_0 is false and computing the rate of rejections.

The following special cases of (1) are equivalent to exact F-test models, in the sense of identical d.f. and noncentrality parameters:

- If in the model $(A \succ \boldsymbol{B}) \times \boldsymbol{C}$ we have $\sigma^2_{\beta(\alpha)} = 0$, then it is equivalent to $(A \times \boldsymbol{C}) \succ \boldsymbol{B}$ and $A \times (\boldsymbol{C} \succ \boldsymbol{B})$; while if $\sigma^2_{\alpha\gamma} = 0$, then it is equivalent to $A \succ \boldsymbol{B} \succ \boldsymbol{C}$.

The Approximate Case (cont.)



Power and size for the mixed model $(A \succ \boldsymbol{B}) \times \boldsymbol{C}$, for $a = 6$, $\alpha = 0.05$, $\delta = 5$, and three variance component assignments $(\sigma^2_{\beta(\alpha)}, \sigma^2_{\gamma}, \sigma^2_{\alpha\gamma}, \sigma^2_{\beta\gamma(\alpha)}, \sigma^2) = (10, 5, 0, 5, 5)$, $(5, 5, 5, 5, 5)$, $(0, 5, 10, 5, 5)$, from left to right. Each contour plot shows the guaranteed power $P_{\min} = (1 - \beta)_{\min}$ (solid curves) overlaid with the size factor $b \cdot c$ (red, dashed hyperbolas) as functions of $b, c \leq 25$, for fixed $n = 2$.

R Programs

We have written several R functions to calculate the power and size:

- power calculation in case an exact F-test exists
  `pwrAnovaExact`
- size calculation in case an exact F-test exists
  `sizeAnovaExact`

A preliminary version of the R package `miniSize` will be available at `http://short.boku.ac.at/iasc-download` soon.

R Programs

Power of the exact F-test:

```
> library(miniSize)

Minimal Size and power calculation of ANOVA models (version 0.3)

> ExistsModel("A>BB>CC")
[1] TRUE

> DsgnCtrl <- AnovaExact.DsgnControl(model="A>BB>CC",
+                                    levels=c(6, 4, 2, 3),
+                                    v=c(bina=1/18, cinab=1/9, e=1/6),
+                                    d=1, case="least")
> res6 <- pwrAnovaExact(DsgnControl=DsgnCtrl)
> res6$pwr
[1] 0.7124785
```

R Programs

Size of the model in case an exact F-test exists:

```
> ##  everything unspecified:
> erg5f <- sizeAnovaExact(model="A>BB>CC", levels=c(6, NA, NA))
> erg5f$random
 a  b  c  n
 6 34  2  2

> sizeAnovaExact(model="A>BB>CC", levels=c(6, NA, NA), v="help")
Error: The vector of variance components has to be a named vector.
The elements should be named by 'bina', 'cinab', and 'e',
e.g., v = c(bina=1, cinab=1, e=1).

> ##  variance components specified:
> erg5b <- sizeAnovaExact(model="A>BB>CC", levels=c(6, NA, NA),
+                         v=c(bina=1/18, cinab=1/9, e=1/6))
> erg5b$random
a b c n
6 7 2 2
```

Summary

We considered balanced ANOVA models up to three factors with at least one fixed factor A.

- We listed the noncentrality parameter for all models.
- We derived a general formula for the worst case noncentrality parameter, required to obtain the guaranteed power of an ANOVA experiment.
- We showed how to determine the minimal experimental size for ANOVA experiments by a new structural result that we call “pivot” effect.
- We provided R functions to calculate the power and size (R package `miniSize`).

References

- Rasch, D., Pilz, J., Verdooren, R., Gebhardt, A. (2011). *Optimal Experimental Design with R*, Chapman & Hall.
- Satterthwaite, F. (1946). An approximate distribution of estimates of the variance components. *Biometrics Bulletin*, 2, 110–114.
- Spangl, B., Kaiblinger, N., Ruckdeschel, P., Rasch D. (2021). Minimal sample size in balanced ANOVA models of crossed, nested, and mixed classifications. *Comm. Stat.—Theory and Methods*, DOI: 10.1080/03610926.2021.1938126.
- Underwood, A.J. (1993). The mechanics of spatially replicated sampling programmes to detect environmental impacts in a variable world. *Australian Journal of Ecology*, 18, 99–116.
- Wille, K., McLean, E., Goddard, J., Byatt, J. (2002). Dietary lipid level and growth hormone alter growth and body conformation of blue tilapia, Oreochromis aureus. *Aquaculture*, 209, 219–232.

Thank you for your attention!