Anomaly Detection in Ordinal Quality-Related Processes by Control Charts

HELMUT SCHMIDT UNIVERSITÄT
Universität der Bundeswehr Hamburg

MATH STAT

Sebastian Ottenstreuer, C.H. Weiß

Department of Mathematics & Statistics,
Helmut Schmidt University, Hamburg

Murat Caner Testik

Department of Industrial Engineering,
Hacettepe University, Ankara

Funded by DFG – Projektnr. 516522977

HELMUT SCHMIDT UNIVERSITÄT
Universität der Bundeswehr Hamburg

MATH STAT

Anomaly Detection by Control Charts

Introduction & Outline

Anomaly Detection by Control Charts

Statistical process control (SPC): (Montgomery, 2009)
monitor quality-related processes, for example,
in manufacturing, service industries, health surveillance.

Control chart: certain statistics computed sequentially in time and used to decide about actual state of process.
Aim: anomaly detection, "deviations from normality".

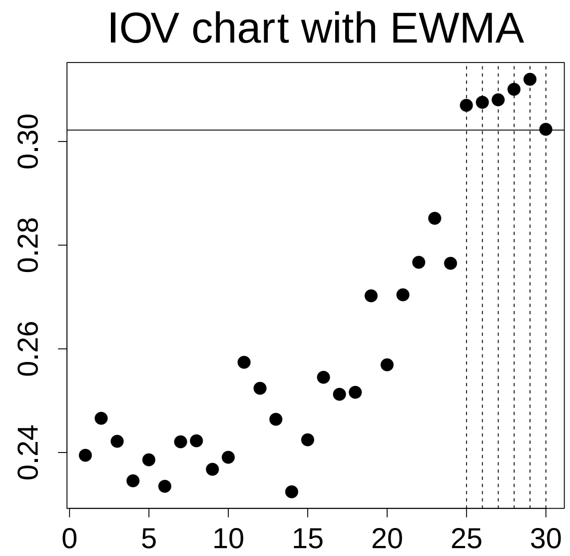
No intervention in process if **in control** (IC), i. e., if monitored statistics stationary according to specified time series model (e. g., independent and identically distributed (i. i. d.) with specified marginal distribution).

By contrast, if deviations from IC-model, such as shifts or drifts in model parameters, then process **out of control** (OOC).

In traditional control chart applications, we compare plotted statistics against given control limits (CLs). If statistic beyond CLs, then alarm triggered to indicate possible OOC-situation.

Example control chart:
(see below)

Alarms at $t \geq 25$, because upper CL violated.



Aim: *trustworthy* anomaly detection, i. e.,

true alarm as soon as possible,

but avoid false alarm for as long as possible.

Common metric: mean waiting time until first alarm,

average run length (ARL) of control chart.

Should be large (low) if process IC (OOC).

In practice: choose CLs such that IC-ARL meets target value.

For these and further basics, see Montgomery (2009).

Most SPC literature about quality characteristics measured

on continuous quantitative scale (**variables charts**).

Here: discrete-valued characteristics, **attributes charts**.

Focus on samples $\{X_{t,i}\}$ of size $n > 1$ from i. i. d. **ordinal process** monitored sequentially in time t.

Quality features $X_{t,i}$ have finite range $\mathcal{S} = \{s_0, s_1, \ldots, s_d\}$ of categories exhibiting natural order $s_0 < \ldots < s_d$.

Data example considered below (Li et al., 2014): manufacturing of electric toothbrush heads, sample size $n = 64$. $X_{t,i}$ = extent of "flash" (excess plastic) in $d + 1 = 4$ ordinal categories $s_0 =$ "slight", $s_1 =$ "small", $s_2 =$ "medium", and $s_3 =$ "large" (degrading quality, as higher risk of injury).

HELMUT SCHMIDT UNIVERSITÄT
Universität der Bundeswehr Hamburg

MATH
STAT

Control Charts for Ordinal Samples

Survey & Proposals

Distribution of ordinal $X_{t,i}$ given by (Agresti, 2010)
probability mass function (PMF), $\boldsymbol{p} = (p_0, \ldots, p_d) \in [0;1]^{d+1}$
with $p_i = P(X = s_i)$, or cumulative distribution function (CDF),
$\boldsymbol{f} = (f_0, \ldots, f_{d-1}) \in [0;1]^d$ with $f_i = P(X \leq s_i)$.

IC-state given by $\boldsymbol{p}_0$ and $\boldsymbol{f}_0$, respectively.

Monitoring uses information provided by tth sample $\{X_{t,i}\}$:
raw or cumulative frequencies,
$\boldsymbol{N}_t = (N_{t,0}, \ldots, N_{t,d})$ resp. $\boldsymbol{C}_t = (C_{t,0}, \ldots, C_{t,d-1})$,
where $N_{t,j}$ ($C_{t,j}$) number of events "$X_{t,i} = s_j$" ("$X_{t,i} \leq s_j$").
Relative (cumulative) frequencies: $\widehat{\boldsymbol{p}}_t = \frac{1}{n}\boldsymbol{N}_t$ and $\widehat{\boldsymbol{f}}_t = \frac{1}{n}\boldsymbol{C}_t$.

Shewhart charts (memory-less) compare $\widehat{\boldsymbol{p}}_t$ to $\boldsymbol{p}_0$,
or $\widehat{\boldsymbol{f}}_t$ to $\boldsymbol{f}_0$, by plotting some function $g(\boldsymbol{N}_t)$ at each time t.

Common approaches for **memory-type charts:**
cumulative sum (CUSUM) of Page (1954), and exponentially weighted moving-average (EWMA) of Roberts (1959).

Ordinal EWMA charts with smoothing param. $\lambda \in (0;1)$:

$$\boldsymbol{N}_t^{(\lambda)} \;=\; \lambda\,\boldsymbol{N}_t + (1-\lambda)\,\boldsymbol{N}_{t-1}^{(\lambda)} \quad \text{for } t = 1, 2, \ldots, \qquad \boldsymbol{N}_0^{(\lambda)} \;=\; n\,\boldsymbol{p}_0.$$

Monitored statistic is $g\big(\boldsymbol{N}_t{}^{(\lambda)}\big)$ for $t = 1, 2, \ldots$,
see Li et al. (2014), Wang et al. (2018) for such examples.

Ordinal Quadratic-Form Statistics

Pearson's goodness-of-fit (GoF) statistic (Duncan, 1950)

$$\mathsf{X}_t^2 \;=\; n^{-1}\big(\boldsymbol{N}_t - n\,\boldsymbol{p}_0\big)^\top \mathrm{diag}(\boldsymbol{p}_0)^{-1}\big(\boldsymbol{N}_t - n\,\boldsymbol{p}_0\big).$$

Average cumulative data (ACD) chart of Wang et al. (2018):

$$\mathrm{ACD}_t \;=\; n^{-1}\textstyle\sum_{j=0}^{d}\Big(C_{t,j-1} + C_{t,j} - n\,(f_{0,j-1} + f_{0,j})\Big)^2.$$

Univ. location-scale ordinal (ULSO) chart of Bai & Li (2021):

$$\mathrm{ULSO}_t \;=\; n^{-1}\big(\boldsymbol{N}_t - n\,\boldsymbol{p}_0\big)^\top \mathbf{V}\,\big(\boldsymbol{N}_t - n\,\boldsymbol{p}_0\big), \qquad \text{with}$$

$$\mathbf{V} \;=\; \mathbf{Q}^\top\Big(\mathbf{Q}\,\big(\mathrm{diag}(\boldsymbol{p}_0) - \boldsymbol{p}_0\,\boldsymbol{p}_0^\top\big)\,\mathbf{Q}^\top\Big)^{-1}\mathbf{Q}, \quad \text{where } \mathbf{Q} = (q_{kl}) \text{ by}$$

$$q_{1j} \;=\; f_{0,j-1} + f_{0,j} - 1, \quad q_{2j} \;=\; p_{0,j}^{-1}\Big(\eta(f_{0,j}) - \eta(f_{0,j-1})\Big),$$

where $\eta(z) = z(1-z)\,\ln\big((1-z)/z\big)$ with $\eta(0) = \eta(1) = 0$.

Statistics based on Weighted Class Counts

Control charts relying on type of weighted class count:

$$D_t \;=\; v_0\, N_{t,0} + \cdots + v_d\, N_{t,d}.$$

If numerical scores $\mathcal{V} = \{v_0, \ldots, v_d\}$ to express severity of defects, then **demerit chart**. Examples:

- Dodge & Torrey (1956): $d+1 = 4$ and scores $1, 10, 50, 100$;
- Nembhard & Nembhard (2000): $d+1 = 3$ and scores $1, 3, 10$;
- Wardell & Candia (1996): scores $1, \ldots, d+1$ (Likert scale).

Weights might also be derived from **probabilistic principles:** (...)

Simple ordinal categorical (SOC) chart of Li et al. (2014),

$$\mathsf{SOC}_t \;=\; \Big|\sum_{j=0}^{d} (f_{0,j-1} + f_{0,j} - 1)\, N_{t,j}\Big| \quad \text{with } f_{0,-1} := 0.$$

If relevant OOC-scenario $\boldsymbol{p}_1$: **log-likelihood ratio** (log-LR)

$$\ell R_t \;=\; \textstyle\sum_{j=0}^{d} N_{t,j}\, \ln\big(p_{1,j}/p_{0,j}\big).$$

Steiner et al. (1996), Ryan et al. (2011): CUSUM chart

$$\mathsf{C}_t \;=\; \max\{0,\, \ell R_t + \mathsf{C}_{t-1}\}, \qquad \mathsf{C}_0 = 0.$$

Shiryaev–Roberts (SR) chart: (Roberts, 1966)

$$\mathsf{R}_t \;=\; (\mathsf{R}_{t-1} + 1)\, \exp\big(\ell R_t\big), \quad \mathsf{R}_0 = 0.$$

Control charts might be based on ordinal statistics, which express important properties of an ordinal X, see Weiß (2020).

- **Dispersion** expressed by index of ordinal variation: IOV chart of Bashkansky & Gadrich (2011), Weiß (2021),

$$\mathrm{IOV}_t \;=\; \tfrac{4}{d} \sum_{j=0}^{d-1} \widehat{f}_{t,j}\,\big(1-\widehat{f}_{t,j}\big).$$

- **Ordinal skewness:** $\mathrm{skew}_t \;=\; \Big(\tfrac{2}{d} \sum_{j=0}^{d-1} \widehat{f}_{t,j}\Big) \;-\; 1,$

  equivalent to demerit chart with linear weighting scheme.

Further miscellaneous approaches in Ottenstreuer et al. (2023).

Helmut Schmidt Universität
Universität der Bundeswehr Hamburg

MATH
STAT

Control Charts for Ordinal Samples

Performance Analysis

Ottenstreuer et al. (2023):

comprehensive comparative **simulation study**,

ARL performance in medium- and high-quality settings.

Summary of main findings:

Although some charts quite sophisticated,

quality deteriorations best detected by rather basic statistics:

demerit-type chart (e. g., skew chart) with EWMA smoothing

always good performance,

(EWMA-)IOV chart for high-quality settings.

EWMA smoothing for PMF estimation generally recommended.

HELMUT SCHMIDT UNIVERSITÄT
Universität der Bundeswehr Hamburg

MATH STAT

Monitoring Flash on Electric Toothbrush Heads

Illustrative Data Example

Toothbrush data (Li et al., 2014),

ranging from $s_0 =$ "slight" to $s_3 =$ "large" flash.

IC-PMF $\boldsymbol{p}_0 =$

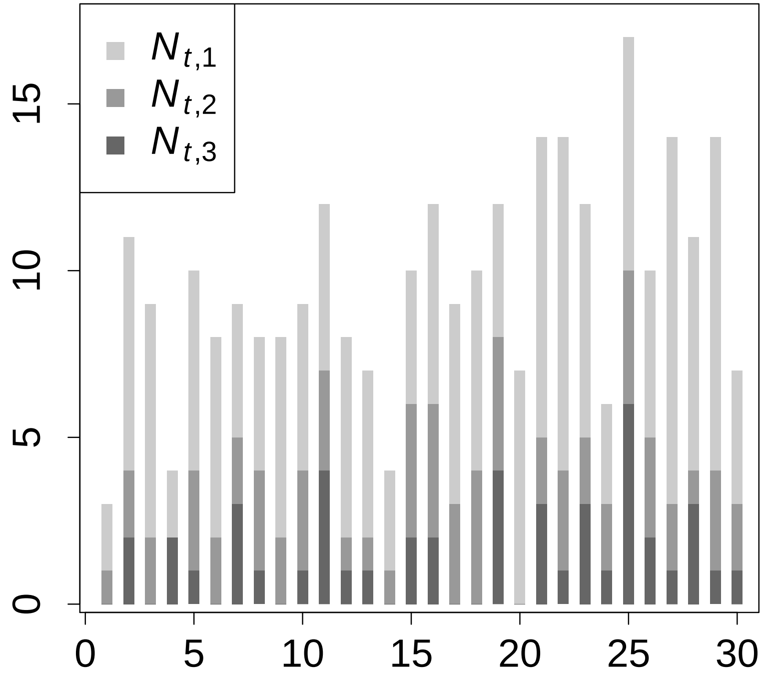
$(0.8631, 0.0804, 0.0357, 0.0208)$.

Frequencies $N_{t,1}, N_{t,2}, N_{t,3}$

of flash types s_1, s_2, s_3

on toothbrush heads, where

$N_{t,0} = 64 - N_{t,1} - N_{t,2} - N_{t,3}$.



All control charts with IC-ARL 370. All **Shewhart charts**

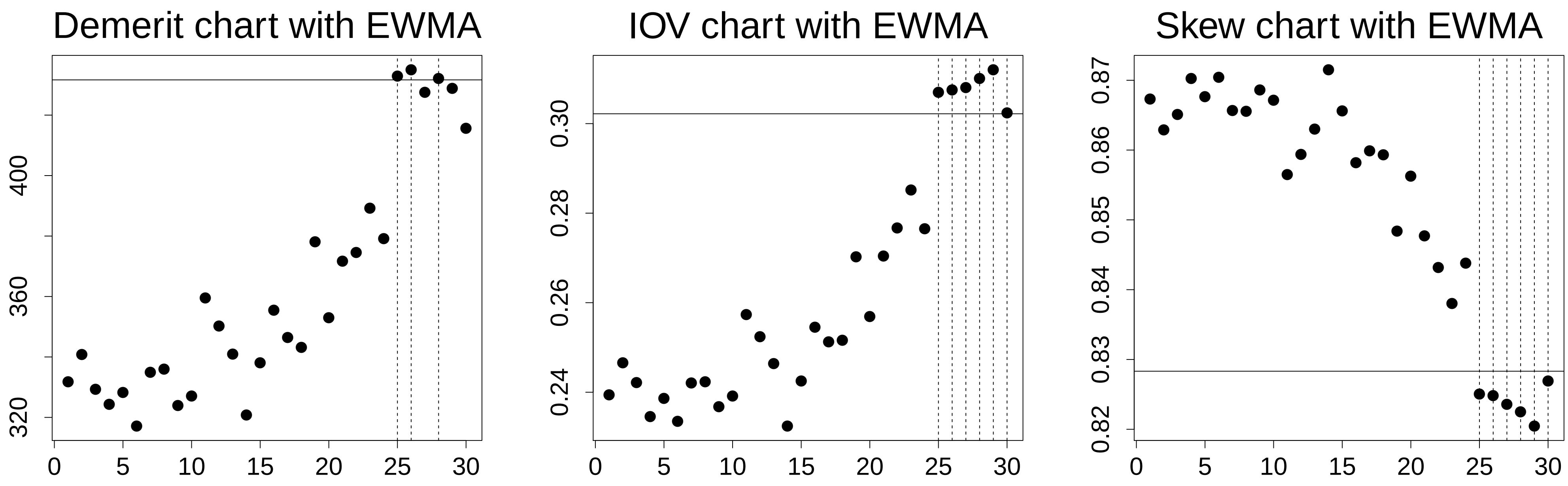
alarm at $t = 25$, where worst quality (see above)

Some EWMA charts with $\lambda = 0.1$ rather slow,

namely Pearson, ACD, ULSO, and SOC.

Also CUSUM and SR chart give late alarm.

Fasted EWMA charts:



Here, demerit chart uses weights $1, 10, 50, 100$.

Conclusions & Outlook

Most surprising finding: For monitoring i. i. d. ordinal samples, simple charts like demerit, skew, IOV EWMA better performance than more sophisticated schemes.

Future research:

DGP \ Data	Samples	Individual Observations
i. i. d. ordinal data	✓ (this talk)	to be done
ordinal time series	to be done	to be done

→ DFG Project No. 516522977, collaboration with Murat C. Testik.

Work in progress:

within DFG Project No. 516522977, jointly with Osama Swidan,

novel models for ordinal time series:

- weighted discrete ARMA models (under review),
- ordinal Hidden-Markov models (under review),
- soft-clipping autoregressive models (in progress).

These and further DGPs to be used for performance analyses of future control charts for ordinal time series data.

Memory-type control charts for individuals data.

Thank You for Your Interest!

Christian H. Weiß

Department of Mathematics & Statistics

Helmut Schmidt University, Hamburg

weissc@hsu-hh.de

HELMUT SCHMIDT UNIVERSITÄT

Universität der Bundeswehr Hamburg

This research was funded by the

Deutsche Forschungsgemeinschaft

(DFG, German Research Foundation),

Projektnummer 516522977.

MATH STAT

Ottenstreuer et al. (2023)
A review and comparison of control charts for ordinal samples.
Journal of Quality Technology **55**(4), 422–441. (→ open access)

Agresti (2010) *Analysis of Ordinal Categorical Data*. 2nd ed., Wiley.

Bai & Li (2021) Location-scale monitoring ... *NRL* **68**, 937–950.

Bashkansky & Gadrich (2011) Statistical quality ... *ASMBI* **27**, 586–599.

Dodge & Torrey (1956) A check inspection and demerit ... *IQC* **13**, 5–12.

Duncan (1950) A chi-square chart for ... *IQC* **7**, 11–15.

Li et al. (2014) A simple categorical chart ... *IJPR* **52**, 550–562.

Montgomery (2009) *Introduction to Statistical Quality ...* 6th ed., Wiley.

Nembhard & Nembhard (2000) A demerits control ... *QE* **13**, 179–190.

Page (1954) Continuous inspection schemes. *Biometrika* **41**, 100–115.

(...)

Literature

(. . .)

Roberts (1959) Control chart tests . . . *Technometrics* **1**, 239–250.

Roberts (1966) A comparison of some control . . . *Technom.* **8**, 411–430.

Ryan et al. (2011) Methods for monitoring . . . *JQT* **43**, 237–248.

Steiner et al. (1996) Grouped data-sequential . . . *Technom.* **38**, 230–237.

Wang et al. (2018) A univariate procedure for . . . *CSSC* **47**, 115–128.

Wardell & Candia (1996) Statistical process monitoring . . . *QMJ* **3**, 36–50.

Weiß (2020) Distance-based analysis . . . *JASA* **115**, 1189–1200.

Weiß (2021) On approaches for monitoring . . . *Control Charts . . .*, 105–129.