Application of Sign Depth to Point Process Models

Dr. Mirko A. Jakubzik

Statistics with Applications
in Engineering Sciences

Department of Statistics

TU Dortmund University

jakubzik@statistik.tu-dortmund.de

Dortmund, May 15, 2024

Load Sharing & Damage Accumulation

- In a **load sharing system**, the failure of one component affects the performance of the remaining components.

  ↪ Whenever a tension wire breaks, the load is redistributed among the remaining wires.

  ↪ The risk of another failure increases.

  ↪ The failure risk depends on the *number* of remaining components.

- If the failure risk also depends on *how long* the components were exposed to the load, the system is subject to **damage accumulation**.

Figure: Broken tension wires.

Load Sharing & Damage Accumulation

- In a **load sharing system**, the failure of one component affects the performance of the remaining components.

  ↪ Whenever a tension wire breaks, the load is redistributed among the remaining wires.

  ↪ The risk of another failure increases.

  ↪ The failure risk depends on the *number* of remaining components.

- If the failure risk also depends on *how long* the components were exposed to the load, the system is subject to **damage accumulation**.

Figure: Broken tension wires.

Notation Used in This Talk

- In a system with $I \in \mathbb{N}$ components, we denote the random failure times by

$$0 < T_1 < T_2 < \ldots < T_I\,, \qquad \text{and } T_i = \infty \text{ for } i > I.$$

- We denote the corresponding realizations by $t_1, \ldots, t_I$, so that $T_i(\omega) = t_i$ $(i \in \mathbb{N})$.
- The counting process $N = (N_t)_{t \geq 0}$ counts the failures up to the time $t \geq 0$,

$$N_t := \sum_{i=1}^{\infty} \mathbb{1}_{(0,t]}(T_i)\,.$$

- For each $j \in \mathbb{N}$ we consider an independent copy of this system:
  - The counting processes $N^{(j)}$, $j \in \mathbb{N}$, are i.i.d. copies of N.
  - Similarly, let $\left(T_i^{(j)}\right)_{i \in \mathbb{N}}$ denote the point process associated with $N^{(j)}$.

Approach for an intensity-based model:

Identify the failure risk of the components in system j with the stochastic intensity of the counting process $N^{(j)}$.

Hazard Function & Stochastic Intensity

- The *conditional hazard function* h_i of $T_i^{(j)}$ given $T_1^{(j)} = t_1^{(j)}, \dots, T_{i-1}^{(j)} = t_{i-1}^{(j)}$ is

conditional density function → f_i

conditional survival function → S_i

$$h_i\left(t \mid t_1^{(j)}, \dots, t_{i-1}^{(j)}\right) = \frac{f_i\left(t \mid t_1^{(j)}, \dots, t_{i-1}^{(j)}\right)}{S_i\left(t \mid t_1^{(j)}, \dots, t_{i-1}^{(j)}\right)} .$$

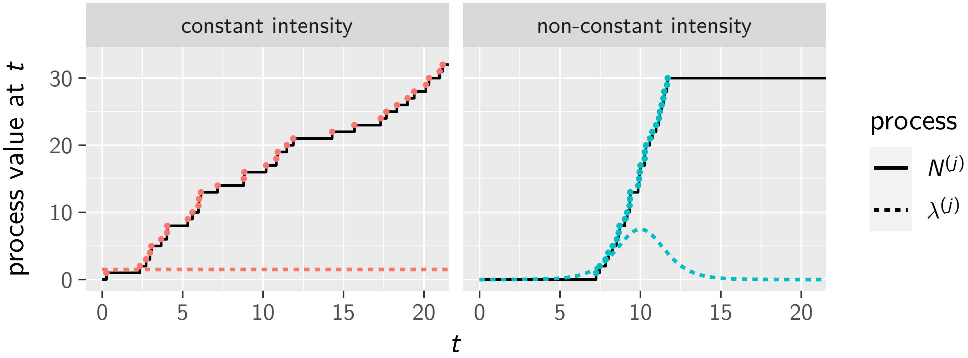
- The *stochastic intensity* $\lambda^{(j)}$ of the counting process $N^{(j)}$ is then piecewise defined by

$$\lambda^{(j)}(t) := \begin{cases} h_1(t)\,, & 0 \le t \le t_1^{(j)}\,, \\ h_i\left(t \mid t_1^{(j)}, \dots, t_{i-1}^{(j)}\right), & t_{i-1}^{(j)} < t \le t_i^{(j)},\ i \ge 2\,. \end{cases}$$

- $\lambda^{(j)}$ constitutes the instantaneous component failure rate of the jth system, that is,

$$\lambda_t^{(j)}\,\mathrm{d}t = \mathbb{E}\left[N^{(j)}(\mathrm{d}t) \,\middle|\, \sigma(\{N_s^{(j)} : s < t\})\right] = \mathbb{P}\left[N^{(j)}(\mathrm{d}t) = 1 \,\middle|\, \sigma(\{N_s^{(j)} : s < t\})\right] .$$

Parametric Intensity-Based Point Process Model



Parametric intensity-based point process model

Let $\theta \in \Theta$ be the parameter of interest, where $\Theta \subset \mathbb{R}^d$, $d \in \mathbb{N}$. An intensity-based point process model is given by a parametric class of intensities,

$$\mathcal{M} = \{\lambda_\theta \,:\, \theta \in \Theta\} \,.$$

We assume the following:

- $\mathcal{M}$ contains the true intensity λ of N.
- There is a *true* $\theta^* \in \Theta$ such that $\lambda = \lambda_{\theta^*}$.

Basquin Load Sharing Model With Damage Accumulation

$$\lambda_\theta^{(j)}(t) := \theta_1 \left(s_j \, \frac{l}{l - N_{t^-}^{(j)}} \right)^{\theta_2} A_j(t)^{\theta_3} \, \mathbb{1}_{\{N_{t^-}^{(j)} < c_j\}} .$$

Figure: Schematic of the Basquin load sharing model with damage accumulation (cf. Basquin 1910, based on Müller and Meyer 2022).

For each $j \in \mathbb{N}$, the damage accumulation term A_j is defined as:

$$A_j(t) = \int_0^t s_j \frac{l}{l - N_{u^-}^{(j)}} \, \mathrm{d}u .$$

Basquin Load Sharing Model With Damage Accumulation

jth system

$$\lambda_\theta^{(j)}(t) := \theta_1 \left(s_j \frac{l}{l - N_{t^-}^{(j)}} \right)^{\theta_2} A_j(t)^{\theta_3} \mathbb{1}_{\{N_{t^-}^{(j)} < c_j\}} .$$

Figure: Schematic of the Basquin load sharing model with damage accumulation (cf. Basquin 1910, based on Müller and Meyer 2022).

For each $j \in \mathbb{N}$, the damage accumulation term A_j is defined as:

$$A_j(t) = \int_0^t s_j \frac{l}{l - N_{u^-}^{(j)}} \, \mathrm{d}u .$$

Basquin Load Sharing Model With Damage Accumulation

jth system

$$\lambda_{\theta}^{(j)}(t) := \theta_1 \left(s_j \frac{l}{l - N_{t^-}^{(j)}} \right)^{\theta_2} A_j(t)^{\theta_3} \, \mathbb{1}_{\{N_{t^-}^{(j)} < c_j\}} \, .$$

model parameter
$\theta = (\theta_1, \theta_2, \theta_3)^\top$

Figure: Schematic of the Basquin load sharing model with damage accumulation (cf. Basquin 1910, based on Müller and Meyer 2022).

For each $j \in \mathbb{N}$, the damage accumulation term A_j is defined as:

$$A_j(t) = \int_0^t s_j \frac{l}{l - N_{u^-}^{(j)}} \, \mathrm{d}u \, .$$

Basquin Load Sharing Model With Damage Accumulation

jth system

initial stress level

$$\lambda^{(j)}_{\theta}(t) := \theta_1 \left(s_j \frac{I}{I - N^{(j)}_{t-}} \right)^{\theta_2} A_j(t)^{\theta_3} \mathbb{1}_{\{N^{(j)}_{t-} < c_j\}} .$$

model parameter

$\theta = (\theta_1, \theta_2, \theta_3)^\top$

Figure: Schematic of the Basquin load sharing model with damage accumulation (cf. Basquin 1910, based on Müller and Meyer 2022).

For each $j \in \mathbb{N}$, the damage accumulation term A_j is defined as:

$$A_j(t) = \int_0^t s_j \frac{I}{I - N^{(j)}_{u-}} \, \mathrm{d}u .$$

Basquin Load Sharing Model With Damage Accumulation

$$\lambda^{(j)}_{\theta}(t) := \theta_1 \left(\underbrace{s_j \frac{I}{I - N^{(j)}_{t^-}}}_{} \right)^{\theta_2} A_j(t)^{\theta_3} \mathbb{1}_{\{N^{(j)}_{t^-} < c_j\}} .$$

jth system

initial stress level

model parameter $\theta = (\theta_1, \theta_2, \theta_3)^\top$

load sharing term:

I = total number of components.

$N^{(j)}_{t^-}$ = # of failed components before time t.

Figure: Schematic of the Basquin load sharing model with damage accumulation (cf. Basquin 1910, based on Müller and Meyer 2022).

For each $j \in \mathbb{N}$, the damage accumulation term A_j is defined as:

$$A_j(t) = \int_0^t s_j \frac{I}{I - N^{(j)}_{u^-}} \,\mathrm{d}u .$$

Basquin Load Sharing Model With Damage Accumulation

$$\lambda^{(j)}_{\theta}(t) := \theta_1 \left(s_j \frac{I}{I - N^{(j)}_{t-}} \right)^{\theta_2} A_j(t)^{\theta_3} \, \mathbb{1}_{\{N^{(j)}_{t-} < c_j\}} .$$

jth system

initial stress level

model parameter $\theta = (\theta_1, \theta_2, \theta_3)^\top$

damage accumulation term

load sharing term:

I = total number of components.

$N^{(j)}_{t-}$ = # of failed components before time t.

Figure: Schematic of the Basquin load sharing model with damage accumulation (cf. Basquin 1910, based on Müller and Meyer 2022).

For each $j \in \mathbb{N}$, the damage accumulation term A_j is defined as:

$$A_j(t) = \int_0^t s_j \frac{I}{I - N^{(j)}_{u-}} \, \mathrm{d}u .$$

Basquin Load Sharing Model With Damage Accumulation

$$\lambda^{(j)}_{\theta}(t) := \theta_1 \left(s_j \frac{I}{I - N^{(j)}_{t-}} \right)^{\theta_2} A_j(t)^{\theta_3} \, \mathbb{1}_{\{N^{(j)}_{t-} < c_j\}} .$$

- jth system
- initial stress level (s_j)
- end of experiment after c_j failures
- model parameter $\theta = (\theta_1, \theta_2, \theta_3)^\top$
- damage accumulation term ($A_j(t)$)
- load sharing term:
  - I = total number of components.
  - $N^{(j)}_{t-}$ = # of failed components before time t.

Figure: Schematic of the Basquin load sharing model with damage accumulation (cf. Basquin 1910, based on Müller and Meyer 2022).

For each $j \in \mathbb{N}$, the damage accumulation term A_j is defined as:

$$A_j(t) = \int_0^t s_j \frac{I}{I - N^{(j)}_{u-}} \, \mathrm{d}u .$$

The K-Sign Depth

Definition (K-sign depth Leckey et al. 2023)

For Residuals $R_n = R_n(\theta)$, the *K-sign depth* is defined as

$$d_K(R_1, \ldots, R_N) := \frac{1}{\binom{N}{K}} \sum_{1 \le i_1 < \ldots < i_K \le N} \mathbb{1}_{\{(R_{i_1}, \ldots, R_{i_K}) \in \mathcal{A}\}} \,.$$

where $\mathcal{A}$ is the set of K-tuples with alternating signs, that is,

$$\mathcal{A} := \left\{ (x_1, \ldots, x_K) \in \mathbb{R}^K \; : \; x_i \cdot x_{i-1} < 0 \text{ for all } i = 2, \ldots, K \right\}.$$

Definition (3-sign depth test for arbitrary hypotheses)

A level $\alpha \in (0,1)$ test for $\mathcal{H}_0 : \theta^* \in \Theta_0 \subset \Theta$ is given via:

$$\text{Reject } \mathcal{H}_0 \text{ if } \sup_{\theta \in \Theta_0} N \left(d_3(R_1(\theta), \ldots, R_N(\theta)) - \frac{1}{4} \right) < q_\alpha \,,$$

where q_α denotes the α-quantile of the asymptotic distribution of the standardized 3-sign depth.

The K-Sign Depth

Definition (K-sign depth Leckey et al. 2023)

For Residuals $R_n = R_n(\theta)$, the *K-sign depth* is defined as

$$d_K\left(R_1, \dots, R_N\right) := \frac{1}{\binom{N}{K}} \sum_{1 \le i_1 < \ldots < i_K \le N} \mathbb{1}_{\left\{\left(R_{i_1}, \ldots, R_{i_K}\right) \in \mathcal{A}\right\}} .$$

where $\mathcal{A}$ is the set of K-tuples with alternating signs, that is,

$$\mathcal{A} := \left\{(x_1, \dots, x_K) \in \mathbb{R}^K \ : \ x_i \cdot x_{i-1} < 0 \text{ for all } i = 2, \dots, K\right\}.$$

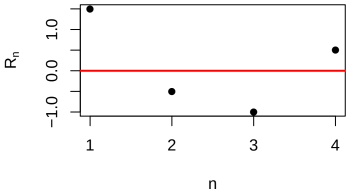
Definition (3-sign depth test for arbitrary hypotheses)

A level $\alpha \in (0, 1)$ test for $\mathcal{H}_0 : \theta^* \in \Theta_0 \subset \Theta$ is given via:

$$\text{Reject } \mathcal{H}_0 \text{ if } \sup_{\theta \in \Theta_0} N\left(d_3(R_1(\theta), \dots, R_N(\theta)) - \frac{1}{4}\right) < q_\alpha \, ,$$

where q_α denotes the α-quantile of the asymptotic distribution of the standardized 3-sign depth.

Example: Let $K = 3$, $N = 4$, $(r_1, \dots, r_4) = (1.5, -0.5, -1, 0.5)$.



alternating : **not altern.**

0 : 0

The K-Sign Depth

Definition (K-sign depth Leckey et al. 2023)

For Residuals $R_n = R_n(\theta)$, the *K-sign depth* is defined as

$$d_K(R_1,\ldots,R_N) := \frac{1}{\binom{N}{K}} \sum_{1\le i_1<\ldots<i_K\le N} \mathbb{1}_{\{(R_{i_1},\ldots,R_{i_K})\in\mathcal{A}\}} .$$

where $\mathcal{A}$ is the set of K-tuples with alternating signs, that is,

$$\mathcal{A} := \left\{(x_1,\ldots,x_K)\in\mathbb{R}^K \;:\; x_i\cdot x_{i-1}<0 \text{ for all } i=2,\ldots,K\right\}.$$

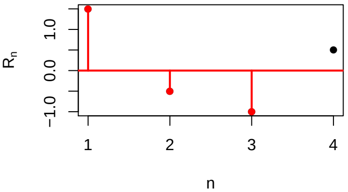
Definition (3-sign depth test for arbitrary hypotheses)

A level $\alpha\in(0,1)$ test for $\mathcal{H}_0 : \theta^*\in\Theta_0\subset\Theta$ is given via:

$$\text{Reject } \mathcal{H}_0 \text{ if } \sup_{\theta\in\Theta_0} N\left(d_3(R_1(\theta),\ldots,R_N(\theta))-\frac{1}{4}\right) < q_\alpha ,$$

where q_α denotes the α-quantile of the asymptotic distribution of the standardized 3-sign depth.

Example: Let $K=3$, $N=4$, $(r_1,\ldots,r_4)=(1.5,-0.5,-1,0.5)$.



alternating	:	not altern.
0	:	1

The K-Sign Depth

Definition (K-sign depth Leckey et al. 2023)

For Residuals $R_n = R_n(\theta)$, the *K-sign depth* is defined as

$$d_K(R_1, \ldots, R_N) := \frac{1}{\binom{N}{K}} \sum_{1 \le i_1 < \ldots < i_K \le N} \mathbb{1}_{\{(R_{i_1}, \ldots, R_{i_K}) \in \mathcal{A}\}} .$$

where $\mathcal{A}$ is the set of K-tuples with alternating signs, that is,

$$\mathcal{A} := \left\{ (x_1, \ldots, x_K) \in \mathbb{R}^K \;:\; x_i \cdot x_{i-1} < 0 \text{ for all } i = 2, \ldots, K \right\}.$$

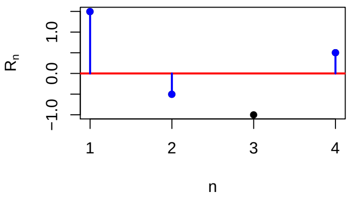
Definition (3-sign depth test for arbitrary hypotheses)

A level $\alpha \in (0,1)$ test for $\mathcal{H}_0 : \theta^* \in \Theta_0 \subset \Theta$ is given via:

$$\text{Reject } \mathcal{H}_0 \text{ if } \sup_{\theta \in \Theta_0} N \left(d_3(R_1(\theta), \ldots, R_N(\theta)) - \frac{1}{4} \right) < q_\alpha ,$$

where q_α denotes the α-quantile of the asymptotic distribution of the standardized 3-sign depth.

Example: Let $K = 3$, $N = 4$, $(r_1, \ldots, r_4) = (1.5, -0.5, -1, 0.5)$.



alternating	:	not altern.
1	:	1

The K-Sign Depth

Definition (K-sign depth Leckey et al. 2023)

For Residuals $R_n = R_n(\theta)$, the *K-sign depth* is defined as

$$d_K\left(R_1, \ldots, R_N\right) := \frac{1}{\binom{N}{K}} \sum_{1 \le i_1 < \ldots < i_K \le N} \mathbb{1}_{\left\{\left(R_{i_1}, \ldots, R_{i_K}\right) \in \mathcal{A}\right\}}.$$

where $\mathcal{A}$ is the set of K-tuples with alternating signs, that is,

$$\mathcal{A} := \left\{(x_1, \ldots, x_K) \in \mathbb{R}^K \; : \; x_i \cdot x_{i-1} < 0 \text{ for all } i = 2, \ldots, K\right\}.$$

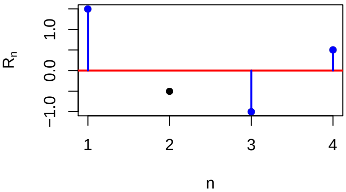
Definition (3-sign depth test for arbitrary hypotheses)

A level $\alpha \in (0,1)$ test for $\mathcal{H}_0 : \theta^* \in \Theta_0 \subset \Theta$ is given via:

$$\text{Reject } \mathcal{H}_0 \text{ if } \sup_{\theta \in \Theta_0} N\left(d_3(R_1(\theta), \ldots, R_N(\theta)) - \frac{1}{4}\right) < q_\alpha\,,$$

where q_α denotes the α-quantile of the asymptotic distribution of the standardized 3-sign depth.

Example: Let $K = 3$, $N = 4$, $(r_1, \ldots, r_4) = (1.5, -0.5, -1, 0.5)$.



alternating	:	**not altern.**
2	:	1

The K-Sign Depth

Definition (K-sign depth Leckey et al. 2023)

For Residuals $R_n = R_n(\theta)$, the *K-sign depth* is defined as

$$d_K\left(R_1, \ldots, R_N\right) := \frac{1}{\binom{N}{K}} \sum_{1 \le i_1 < \ldots < i_K \le N} \mathbb{1}_{\left\{\left(R_{i_1}, \ldots, R_{i_K}\right) \in \mathcal{A}\right\}}.$$

where $\mathcal{A}$ is the set of K-tuples with alternating signs, that is,

$$\mathcal{A} := \left\{(x_1, \ldots, x_K) \in \mathbb{R}^K \;:\; x_i \cdot x_{i-1} < 0 \text{ for all } i = 2, \ldots, K\right\}.$$

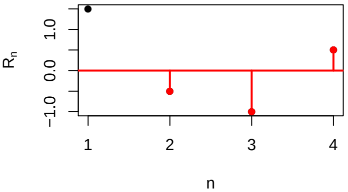
Definition (3-sign depth test for arbitrary hypotheses)

A level $\alpha \in (0, 1)$ test for $\mathcal{H}_0 : \theta^* \in \Theta_0 \subset \Theta$ is given via:

$$\text{Reject } \mathcal{H}_0 \text{ if } \sup_{\theta \in \Theta_0} N\left(d_3(R_1(\theta), \ldots, R_N(\theta)) - \frac{1}{4}\right) < q_\alpha,$$

where q_α denotes the α-quantile of the asymptotic distribution of the standardized 3-sign depth.

Example: Let $K = 3$, $N = 4$, $(r_1, \ldots, r_4) = (1.5, -0.5, -1, 0.5)$.



alternating	:	not altern.
2	:	2

The K-Sign Depth

Definition (K-sign depth Leckey et al. 2023)

For Residuals $R_n = R_n(\theta)$, the *K-sign depth* is defined as

$$d_K(R_1, \ldots, R_N) := \frac{1}{\binom{N}{K}} \sum_{1 \le i_1 < \ldots < i_K \le N} \mathbb{1}_{\{(R_{i_1}, \ldots, R_{i_K}) \in \mathcal{A}\}} .$$

where $\mathcal{A}$ is the set of K-tuples with alternating signs, that is,

$$\mathcal{A} := \left\{(x_1, \ldots, x_K) \in \mathbb{R}^K \;:\; x_i \cdot x_{i-1} < 0 \text{ for all } i = 2, \ldots, K\right\}.$$

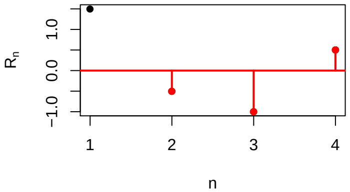
Definition (3-sign depth test for arbitrary hypotheses)

A level $\alpha \in (0,1)$ test for $\mathcal{H}_0 : \theta^* \in \Theta_0 \subset \Theta$ is given via:

$$\text{Reject } \mathcal{H}_0 \text{ if } \sup_{\theta \in \Theta_0} N \left(d_3(R_1(\theta), \ldots, R_N(\theta)) - \frac{1}{4} \right) < q_\alpha ,$$

where q_α denotes the α-quantile of the asymptotic distribution of the standardized 3-sign depth.

Example: Let $K = 3$, $N = 4$, $(r_1, \ldots, r_4) = (1.5, -0.5, -1, 0.5)$.



alternating : **not altern.**

2 : 2

$$d_3(r_1, \ldots, r_4) = \frac{2}{2+2}$$

The K-Sign Depth

Definition (K-sign depth Leckey et al. 2023)

For Residuals $R_n = R_n(\theta)$, the *K-sign depth* is defined as

$$d_K(R_1, \ldots, R_N) := \frac{1}{\binom{N}{K}} \sum_{1 \le i_1 < \ldots < i_K \le N} \mathbb{1}_{\{(R_{i_1}, \ldots, R_{i_K}) \in \mathcal{A}\}} \,.$$

where $\mathcal{A}$ is the set of K-tuples with alternating signs, that is,

$$\mathcal{A} := \left\{ (x_1, \ldots, x_K) \in \mathbb{R}^K \ : \ x_i \cdot x_{i-1} < 0 \text{ for all } i = 2, \ldots, K \right\}.$$

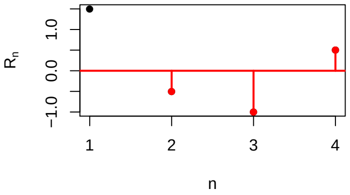
Definition (3-sign depth test for arbitrary hypotheses)

A level $\alpha \in (0,1)$ test for $\mathcal{H}_0 : \theta^* \in \Theta_0 \subset \Theta$ is given via:

$$\text{Reject } \mathcal{H}_0 \text{ if } \sup_{\theta \in \Theta_0} N \left(d_3(R_1(\theta), \ldots, R_N(\theta)) - \frac{1}{4} \right) < q_\alpha \,,$$

where q_α denotes the α-quantile of the asymptotic distribution of the standardized 3-sign depth.

Example: Let $K = 3$, $N = 4$, $(r_1, \ldots, r_4) = (1.5, -0.5, -1, 0.5)$.



alternating : **not altern.**

2 : 2

$$d_3(r_1, \ldots, r_4) = \frac{2}{2 + 2}$$

(K1) $R_1(\theta^*), \ldots, R_N(\theta^*)$ are independent w.r.t. $\mathbb{P}_{\theta^*}$.

(K2) $\mathbb{P}_{\theta^*}\left(R_n(\theta^*) > 0\right) = 1/2$ and $\mathbb{P}_{\theta^*}\left(R_n(\theta^*) < 0\right) = 1/2$.

The Hazard Transform

Definition (Hazard transform)

The hazard transform $\tilde{R}_{j,i}(\theta)$ of $T_i^{(j)}$ at $\theta \in \Theta$ is defined as

$$\tilde{R}_{j,i}(\theta) := H_i^\theta\left(T_i^{(j)} \mid T_1^{(j)}, \ldots, T_{i-1}^{(j)}\right), \quad j \in \mathbb{N}, \, i = 1, \ldots, c_j,$$

where the cumulative conditional hazard function is given by

$$H_i^\theta\left(t \mid T_1^{(j)}, \ldots, T_{i-1}^{(j)}\right) := \int_{T_{i-1}^{(j)}}^{t} h_i^\theta\left(u \mid T_1^{(j)}, \ldots, T_{i-1}^{(j)}\right) \mathrm{d}u.$$

At the true parameter θ^* holds: $\tilde{R}_{j,i}(\theta^*) \overset{\text{i.i.d.}}{\sim} \mathcal{E}(1)$.

Definition (Median-centred hazard transform)

The median-centred hazard transform $R_{j,i}(\theta)$ is defined as

$$R_{j,i}(\theta) := \tilde{R}_{j,i}(\theta) - \ln(2).$$

Reminder:

$\mathcal{M} = \{\lambda_\theta : \theta \in \Theta\}$,

on $\left\{T_{i-1}^{(j)} < t \leq T_i^{(j)}\right\}$:

$\lambda_\theta^{(j)}(t) = h_i^\theta\left(t \mid T_{1:(i-1)}^{(j)}\right)$.

Requirements for the 3-sign depth test:

The Hazard Transform

Definition (Hazard transform)

The hazard transform $\tilde{R}_{j,i}(\theta)$ of $T_i^{(j)}$ at $\theta \in \Theta$ is defined as

$$\tilde{R}_{j,i}(\theta) := H_i^\theta\left(T_i^{(j)} \mid T_1^{(j)}, \dots, T_{i-1}^{(j)}\right), \quad j \in \mathbb{N},\ i = 1, \dots, c_j,$$

where the cumulative conditional hazard function is given by

$$H_i^\theta\left(t \mid T_1^{(j)}, \dots, T_{i-1}^{(j)}\right) := \int_{T_{i-1}^{(j)}}^t h_i^\theta\left(u \mid T_1^{(j)}, \dots, T_{i-1}^{(j)}\right) \mathrm{d}u.$$

At the true parameter θ^* holds: $\tilde{R}_{j,i}(\theta^*) \overset{\text{i.i.d.}}{\sim} \mathcal{E}(1)$.

Definition (Median-centred hazard transform)

The median-centred hazard transform $R_{j,i}(\theta)$ is defined as

$\mathrm{med}(\tilde{R}_{j,i}(\theta^*)) = \ln(2)$

$$R_{j,i}(\theta) := \tilde{R}_{j,i}(\theta) - \ln(2).$$

Reminder:

$\mathcal{M} = \{\lambda_\theta : \theta \in \Theta\}$,

on $\left\{T_{i-1}^{(j)} < t \le T_i^{(j)}\right\}$:

$\lambda_\theta^{(j)}(t) = h_i^\theta\left(t \mid T_{1:(i-1)}^{(j)}\right)$.

Requirements for the 3-sign depth test:

The Hazard Transform

Definition (Hazard transform)

The hazard transform $\tilde{R}_{j,i}(\theta)$ of $T_i^{(j)}$ at $\theta \in \Theta$ is defined as

$$\tilde{R}_{j,i}(\theta) := H_i^{\theta}\left(T_i^{(j)} \mid T_1^{(j)}, \ldots, T_{i-1}^{(j)}\right), \quad j \in \mathbb{N}, \, i = 1, \ldots, c_j,$$

where the cumulative conditional hazard function is given by

$$H_i^{\theta}\left(t \mid T_1^{(j)}, \ldots, T_{i-1}^{(j)}\right) := \int_{T_{i-1}^{(j)}}^{t} h_i^{\theta}\left(u \mid T_1^{(j)}, \ldots, T_{i-1}^{(j)}\right) \mathrm{d}u.$$

At the true parameter θ^* holds: $\tilde{R}_{j,i}(\theta^*) \overset{\text{i.i.d.}}{\sim} \mathcal{E}(1)$.

Reminder:

$\mathcal{M} = \{\lambda_\theta : \theta \in \Theta\}$,

on $\left\{T_{i-1}^{(j)} < t \leq T_i^{(j)}\right\}$:

$\lambda_\theta^{(j)}(t) = h_i^\theta\left(t \mid T_{1:(i-1)}^{(j)}\right)$.

Requirements for the 3-sign depth test:

The transforms are independent.
$\Rightarrow$ (K1) is met.

Definition (Median-centred hazard transform)

The median-centred hazard transform $R_{j,i}(\theta)$ is defined as

$\operatorname{med}\left(\tilde{R}_{j,i}(\theta^*)\right) = \ln(2)$

$$R_{j,i}(\theta) := \tilde{R}_{j,i}(\theta) - \ln(2).$$

The Hazard Transform

Definition (Hazard transform)

The hazard transform $\tilde{R}_{j,i}(\theta)$ of $T_i^{(j)}$ at $\theta \in \Theta$ is defined as

$$\tilde{R}_{j,i}(\theta) := H_i^\theta\left(T_i^{(j)} \mid T_1^{(j)}, \ldots, T_{i-1}^{(j)}\right), \quad j \in \mathbb{N}, \, i = 1, \ldots, c_j,$$

where the cumulative conditional hazard function is given by

$$H_i^\theta\left(t \mid T_1^{(j)}, \ldots, T_{i-1}^{(j)}\right) := \int_{T_{i-1}^{(j)}}^t h_i^\theta\left(u \mid T_1^{(j)}, \ldots, T_{i-1}^{(j)}\right) \mathrm{d}u.$$

At the true parameter θ^* holds: $\tilde{R}_{j,i}(\theta^*) \overset{\text{i.i.d.}}{\sim} \mathcal{E}(1)$.

Definition (Median-centred hazard transform)

The median-centred hazard transform $R_{j,i}(\theta)$ is defined as

$\mathrm{med}(\tilde{R}_{j,i}(\theta^*)) = \ln(2)$

$$R_{j,i}(\theta) := \tilde{R}_{j,i}(\theta) - \ln(2).$$

Reminder:

$\mathcal{M} = \{\lambda_\theta \,:\, \theta \in \Theta\}$,

on $\left\{T_{i-1}^{(j)} < t \leq T_i^{(j)}\right\}$:

$\lambda_\theta^{(j)}(t) = h_i^\theta\left(t \mid T_{1:(i-1)}^{(j)}\right)$.

Requirements for the 3-sign depth test:

The transforms are independent. $\Rightarrow$ (K1) is met.

Transforms are median-centred. $\Rightarrow$ (K2) is met.

The Hazard Transform

Definition (Hazard transform)

The hazard transform $\tilde{R}_{j,i}(\theta)$ of $T_i^{(j)}$ at $\theta \in \Theta$ is defined as

$$\tilde{R}_{j,i}(\theta) := H_i^\theta\left(T_i^{(j)} \mid T_1^{(j)}, \ldots, T_{i-1}^{(j)}\right), \quad j \in \mathbb{N}, \, i = 1, \ldots, c_j \,,$$

where the cumulative conditional hazard function is given by

$$H_i^\theta\left(t \mid T_1^{(j)}, \ldots, T_{i-1}^{(j)}\right) := \int_{T_{i-1}^{(j)}}^{t} h_i^\theta\left(u \mid T_1^{(j)}, \ldots, T_{i-1}^{(j)}\right) \mathrm{d}u \,.$$

At the true parameter θ^* holds: $\tilde{R}_{j,i}(\theta^*) \overset{\text{i.i.d.}}{\sim} \mathcal{E}(1)\,.$

Definition (Median-centred hazard transform)

The median-centred hazard transform $R_{j,i}(\theta)$ is defined as

$\operatorname{med}(\tilde{R}_{j,i}(\theta^*)) = \ln(2)$

$$R_{j,i}(\theta) := \tilde{R}_{j,i}(\theta) - \ln(2)\,.$$

Reminder:

$\mathcal{M} = \{\lambda_\theta \,:\, \theta \in \Theta\}$,

on $\left\{T_{i-1}^{(j)} < t \leq T_i^{(j)}\right\}$:

$\lambda_\theta^{(j)}(t) = h_i^\theta\left(t \mid T_{1:(i-1)}^{(j)}\right)$.

Requirements for the 3-sign depth test:

The transforms are independent. $\Rightarrow$ (K1) is met.

Use 3-sign depth test.

Transforms are median-centred. $\Rightarrow$ (K2) is met.

Robustness of the 3-Sign Depth Test

- In a simulation study, we compare the 3-sign depth test with two other methods:
  - Wald-type test based on a minimum distance estimator of Kopperschmidt and Stute 2013.
  - Likelihood-ratio test constructed from the Cox partial likelihood.
- To assess their robustness, we apply them to contaminated data.
- During the simulation of the data, a fixed proportion p of the data is contaminated:
  1. Determine which $\lfloor p \cdot c_j \rfloor$ waiting times of the jth system are contaminated.
  2. Randomly replace these waiting times with atypically small or large values.

	model parameter		covariates			# of processes
parameter/covariate	θ^*	I	c_j	p	s_j	J
chosen value(s)	$(10^{-4}, 3, 1)^\top$	25	10	0.4	$80, 120, 200$	$9, 90$

Table: The selected model parameters and values of covariates for the simulation study.

Robustness: Type I Errors for Contaminated Data

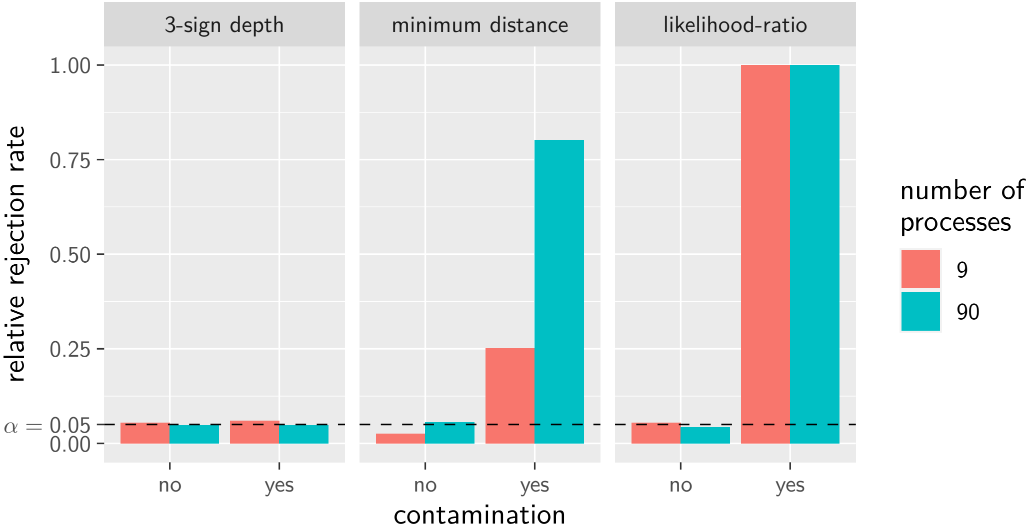


Figure: Type I errors of different level $\alpha = 0.05$ tests for $\mathcal{H}_0 : \theta^* = \theta_0$ with contaminated data.

Robustness: Power of the 3-Sign Depth Test for Contaminated Data

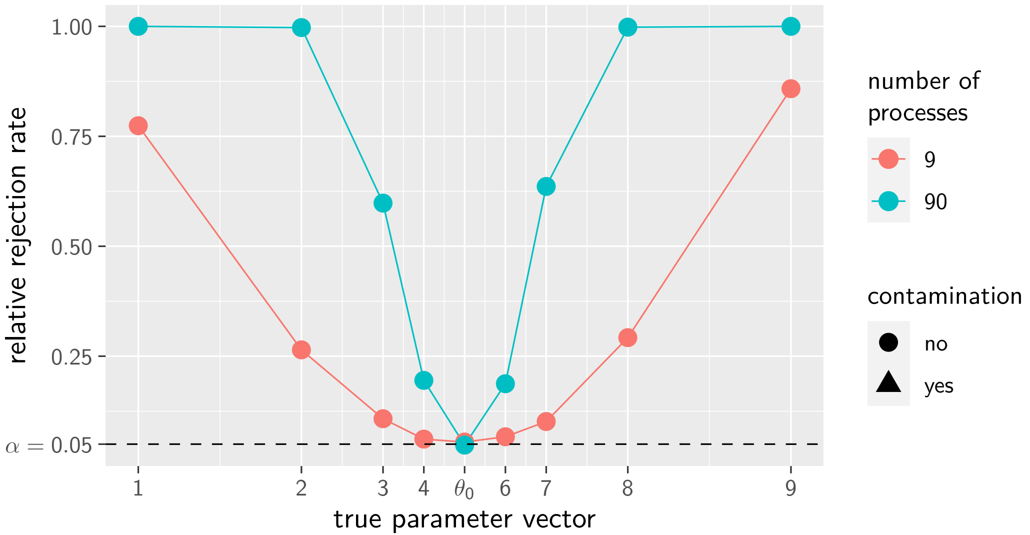


Figure: Power of the 3-sign depth tests for $\mathcal{H}_0 : \theta^* = \theta_0$ with 9 different true parameter vectors along a line through $\theta_0 = \left(10^{-4}, 3, 1\right)^\top$. The scaling on the x-axis matches the actual distances.

Robustness: Power of the 3-Sign Depth Test for Contaminated Data

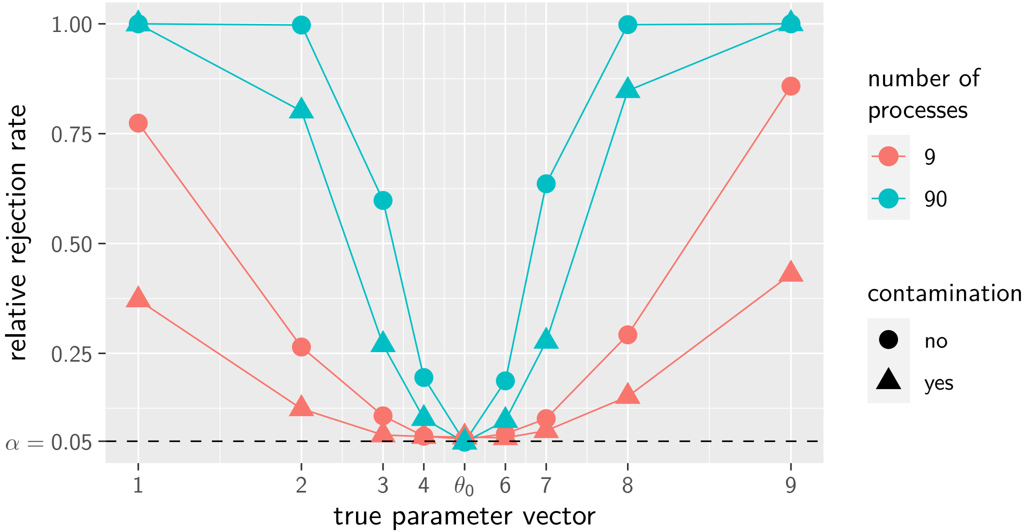


Figure: Power of the 3-sign depth tests for $\mathcal{H}_0 : \theta^* = \theta_0$ with 9 different true parameter vectors along a line through $\theta_0 = \left(10^{-4}, 3, 1\right)^\top$. The scaling on the x-axis matches the actual distances.

Real Data Application: Fatigue Tests

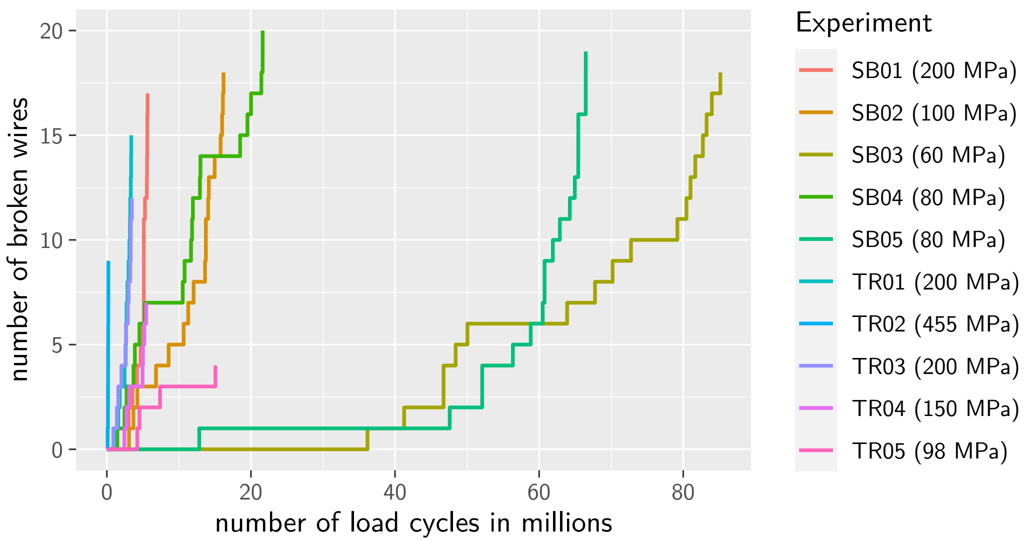


Figure: Results of a series of fatigue tests carried out at TU Dortmund University (Szugat et al. 2016).

Real Data Application: Confidence Set for the True Parameter

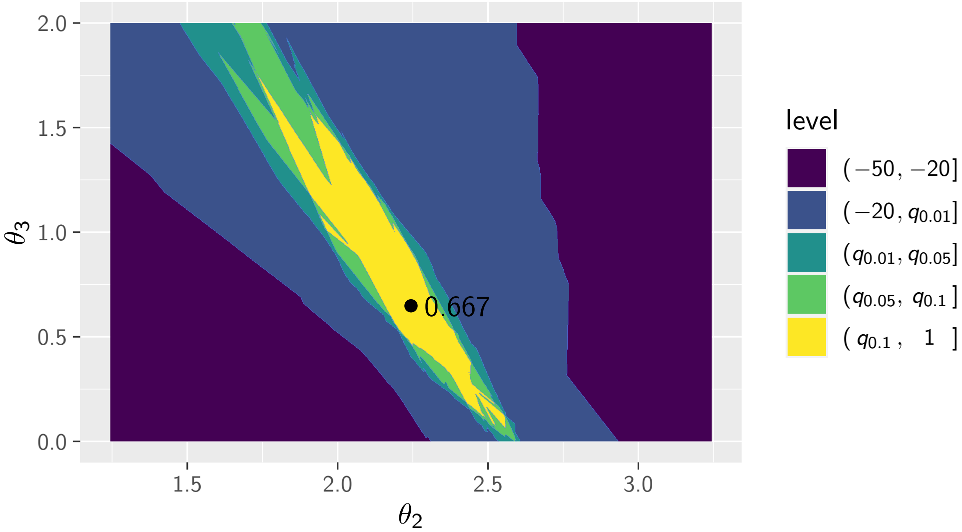


Figure: Two-dimensional slice in θ_2-θ_3-direction of the confidence region for the true parameter θ^* based on the fatigue data. The maximum depth of 0.667 is attained at $(\exp(-25.981), 2.244, 0.648)^\top$.

References I

[1] Odd Aalen. "Nonparametric Inference for a Family of Counting Processes". In: *The Annals of Statistics* 6.4 (1978), pp. 701 –726. DOI: 10.1214/aos/1176344247.

[2] P. K. Andersen and R. D. Gill. "Cox's Regression Model for Counting Processes: A Large Sample Study". In: *The Annals of Statistics* 10.4 (1982), pp. 1100–1120.

[3] Narayanaswamy Balakrishnan, Eric Beutner, and Udo Kamps. "Modeling Parameters of a Load-Sharing System Through Link Functions in Sequential Order Statistics Models and Associated Inference". In: *IEEE Transactions on Reliability* 60.3 (2011), pp. 605–611. DOI: 10.1109/TR.2011.2161152.

[4] Olin Hanson Basquin. "The exponential law of endurance tests". In: *American Society of Testing Materials* 10 (1910), pp. 625–630.

[5] D. R. Cox. "Regression Models and Life-Tables". In: *Journal of the Royal Statistical Society. Series B (Methodological)* 34.2 (1972), pp. 187–220.

[6] D. J. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes. Volume I: Elementary Theory and Methods*. Probability and Its Applications. New York, Berlin, and Heidelberg: Springer-Verlag, 2003. ISBN: 0-387-95541-0.

[7] Kai Kopperschmidt and Winfried Stute. "The Statistical Analysis of Self-Exciting Point Processes". In: *Statistica Sinica* 23 (2013), pp. 1273–1298.

References II

[8] Paul H. Kvam and Edsel A. Peña. "Estimating Load-Sharing Properties in a Dynamic Reliability System". In: *Journal of the American Statistical Association* 100.469 (2005), pp. 262–272.

[9] Kevin Leckey et al. "Prediction intervals for load-sharing systems in accelerated life testing". In: *Quality and Reliability Engineering International* 36.6 (2020), pp. 1895–1915. DOI: https://doi.org/10.1002/qre.2664.

[10] Kevin Leckey et al. "Simple powerful robust tests based on sign depth". In: *Statistical Papers* 64 (2023), pp. 857–882.

[11] Christine H. Müller and Renate Meyer. "Inference of Intensity-Based Models for Load-Sharing Systems With Damage Accumulation". In: *IEEE Transactions on Reliability* 71.2 (2022), pp. 539–554. DOI: 10.1109/TR.2022.3140483.

[12] Fabio L. Spizzichino. "Reliability, signature, and relative quality functions of systems under time-homogeneous load-sharing models". In: *Applied Stochastic Models in Business and Industry* 35.2 (2019), pp. 158–176. DOI: https://doi.org/10.1002/asmb.2397.

[13] Sebastian Szugat et al. "Prediction Intervals for the Failure Time of Prestressed Concrete Beams". In: *Advances in Materials Science and Engineering* 2016 (2016). Article ID 9605450.

[14] Jianchun Zhang, Yu Zhao, and Xiaobing Ma. "Reliability modeling methods for load-sharing k-out-of-n system subject to discrete external load". In: *Reliability Engineering & System Safety* 193 (2020), p. 106603. ISSN: 0951-8320. DOI: https://doi.org/10.1016/j.ress.2019.106603.

Intensitätsbasierte Modelle in der Literatur

- Intensitätsbasierte Modelle finden sich in der Literatur seit den 70er Jahren.
  - Multiplikatives Intensitätsmodell (Aalen 1978)
  - Cox-Regressionsmodell (Cox 1972)
  - Relatives Risikoregressionsmodell (Andersen and Gill 1982)
- Anwendung auf Lastumverteilungssysteme hauptsächlich in den letzten zwanzig Jahren.

  ↪ Kvam and Peña 2005, Balakrishnan, Beutner, and Kamps 2011, Spizzichino 2019, Zhang, Zhao, and Ma 2020, Leckey et al. 2020
- Leckey et al. 2020: Lastumverteilungssystem im relativen Risikoregressionsmodell,

$$h_i^\theta(t \mid T_1^{(j)}, \ldots, T_{i-1}^{(j)}) = \theta_1 \left(s_j \frac{I}{I-(i-1)} \right)^{\theta_2}, \quad i = 1, \ldots, c_j,$$

mit zufälligen Kovariaten $s_j > 0$ und $c_j \in \{1, \ldots, I\}$.

Intensitätsbasierte Modelle in der Literatur

- Intensitätsbasierte Modelle finden sich in der Literatur seit den 70er Jahren.
  - Multiplikatives Intensitätsmodell (Aalen 1978)
  - Cox-Regressionsmodell (Cox 1972)
  - Relatives Risikoregressionsmodell (Andersen and Gill 1982)
- Anwendung auf Lastumverteilungssysteme hauptsächlich in den letzten zwanzig Jahren.

  ↪ Kvam and Peña 2005, Balakrishnan, Beutner, and Kamps 2011, Spizzichino 2019, Zhang, Zhao, and Ma 2020, Leckey et al. 2020
- Leckey et al. 2020: Lastumverteilungssystem im relativen Risikoregressionsmodell,

$$h_i^{\theta}(t \mid T_1^{(j)}, \dots, T_{i-1}^{(j)}) = \theta_1 \left(s_j \frac{I}{I - (i-1)} \right)^{\theta_2}, \quad i = 1, \dots, c_j,$$

mit zufälligen Kovariaten $s_j > 0$ und $c_j \in \{1, \dots, I\}$.

Intensitätsbasierte Modelle in der Literatur

- Intensitätsbasierte Modelle finden sich in der Literatur seit den 70er Jahren.
  - Multiplikatives Intensitätsmodell (Aalen 1978)
  - Cox-Regressionsmodell (Cox 1972)
  - Relatives Risikoregressionsmodell (Andersen and Gill 1982)
- Anwendung auf Lastumverteilungssysteme hauptsächlich in den letzten zwanzig Jahren.

  $\hookrightarrow$ Kvam and Peña 2005, Balakrishnan, Beutner, and Kamps 2011, Spizzichino 2019, Zhang, Zhao, and Ma 2020, Leckey et al. 2020
- Leckey et al. 2020: Lastumverteilungssystem im relativen Risikoregressionsmodell,

$$h_i^{\theta}(t \mid T_1^{(j)}, \ldots, T_{i-1}^{(j)}) = \theta_1 \left(s_j \frac{I}{I-(i-1)} \right)^{\theta_2}, \quad i = 1, \ldots, c_j,$$

mit zufälligen Kovariaten $s_j > 0$ und $c_j \in \{1, \ldots, I\}$.

Intensitätsbasierte Modelle in der Literatur

- Intensitätsbasierte Modelle finden sich in der Literatur seit den 70er Jahren.
  - Multiplikatives Intensitätsmodell (Aalen 1978)
  - Cox-Regressionsmodell (Cox 1972)
  - Relatives Risikoregressionsmodell (Andersen and Gill 1982)
- Anwendung auf Lastumverteilungssysteme hauptsächlich in den letzten zwanzig Jahren.

  ↪ Kvam and Peña 2005, Balakrishnan, Beutner, and Kamps 2011, Spizzichino 2019, Zhang, Zhao, and Ma 2020, Leckey et al. 2020
- Leckey et al. 2020: Lastumverteilungssystem im relativen Risikoregressionsmodell,

$$h_i^{\theta}(t \mid T_1^{(j)}, \ldots, T_{i-1}^{(j)}) = \theta_1 \left(s_j \frac{I}{I-(i-1)} \right)^{\theta_2}, \quad i = 1, \ldots, c_j,$$

mit zufälligen Kovariaten $s_j > 0$ und $c_j \in \{1, \ldots, I\}$.

initial stress level (s_j)

Anzahl beobachtbarer Komponentenausfälle (c_j)

Intensitätsbasierte Modelle in der Literatur

- Intensitätsbasierte Modelle finden sich in der Literatur seit den 70er Jahren.
  - Multiplikatives Intensitätsmodell (Aalen 1978)
  - Cox-Regressionsmodell (Cox 1972)
  - Relatives Risikoregressionsmodell (Andersen and Gill 1982)

Erinnerung:

Auf $\{T_{i-1}^{(j)} < t \leq T_i^{(j)}\}$ gilt:

$\lambda_\theta^{(j)}(t) = h_i^\theta(t \mid T_1^{(j)}, \dots, T_{i-1}^{(j)})$

- Anwendung auf Lastumverteilungssysteme hauptsächlich in den letzten zwanzig Jahren.

  ↪ Kvam and Peña 2005, Balakrishnan, Beutner, and Kamps 2011, Spizzichino 2019, Zhang, Zhao, and Ma 2020, Leckey et al. 2020

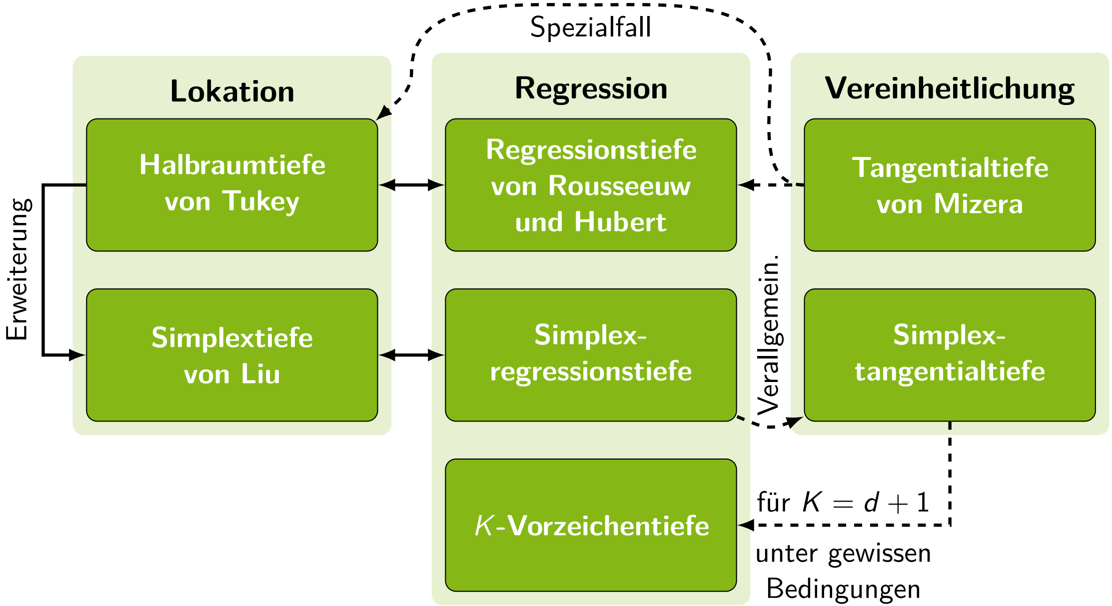
- Leckey et al. 2020: Lastumverteilungssystem im relativen Risikoregressionsmodell,

$$h_i^\theta(t \mid T_1^{(j)}, \dots, T_{i-1}^{(j)}) = \theta_1 \left(s_j \frac{I}{I-(i-1)} \right)^{\theta_2}, \quad i = 1, \dots, c_j,$$

mit zufälligen Kovariaten $s_j > 0$ und $c_j \in \{1, \dots, I\}$.

initial stress level (s_j) — Anzahl beobachtbarer Komponentenausfälle (c_j)

Übersicht über Zusammenhänge verschiedener Tiefekonzepte



Simulation von Punktprozessen

Algorithmus Simulation eines Punktprozesses mit gegebenen kumulierten bedingten Hazardfunktionen $H_i(\cdot \mid t_{1:(i-1)}, x)$ durch die Inversionsmethode, s. Daley and Vere-Jones 2003, p. 260. Erfordert, dass die inverse kum. bedingte Hazardfunktion explizit angegeben werden kann.

Eingabe:

$n \in \mathbb{N}$	Anzahl der zu simulierenden Punkte,
$\mathbb{P}^X$	Verteilung zufälliger Kovariaten,
$t_0 \in \mathbb{R}_+$	Wert von T_0 (determ.), standardmäßig $t_0 = 0$,
$H_i^{-1}(\cdot \mid t_{i-1}, \ldots, t_0, x)$	inverse kum. bedingte Hazardfunktion, $i = 1, \ldots, n$.

Ausgabe:

$t_{1:n} \in \mathbb{R}_+^n$	Vektor der Realisierungen der Punkte $T_1, \ldots, T_n$.

ziehe Zufallsstichprobe x aus der Kovariaten-Verteilung $\mathbb{P}^X$
ziehe u.i.v. Zufallsstichproben $y_1, \ldots, y_n$ aus der Exponentialverteilung $\mathcal{E}(1)$ mit $\lambda = 1$
for $i = 1, \ldots, n$ **do**
    $t_i \leftarrow H_i^{-1}(y_i \mid t_{i-1}, \ldots, t_0, x)$
end for

Verwendete Parameter in der Simulationsstudie

Nummer des Parametervektors	θ_1^*	θ_2^*	θ_2^*
1	$9.2 \cdot 10^{-5}$	2.92	0.92
2	$9.4 \cdot 10^{-5}$	2.96	0.96
3	$9.8 \cdot 10^{-5}$	2.98	0.98
4	$9.9 \cdot 10^{-5}$	2.99	0.99
5	$10.0 \cdot 10^{-5}$	3.00	1.00
6	$10.1 \cdot 10^{-5}$	3.01	1.01
7	$10.2 \cdot 10^{-5}$	3.02	1.02
8	$10.4 \cdot 10^{-5}$	3.04	1.04
9	$10.8 \cdot 10^{-5}$	3.08	1.08

Table: In der Robustheitsstudie verwendete Parametervektoren. θ_0 ist farblich hervorgehoben.

Kontamination der Daten

- In der Robustheitsstudie wurden zwei Arten von Kontamination untersucht.
- Dazu werden Rohdaten modifiziert, d.h. die u.i.v. Zufallsstichproben $y_1, \ldots, y_n$ der $\mathcal{E}(1)$.

1. **Tiefe-spezifische Kontamination:**
   - Kontaminiere Daten durch Erhöhung der Abweichung vom Median ln(2).
   - Für eine Kontamination der *i*ten Beobachtung, ersetze y_i durch

   $$\tilde{y}_i = \max\left\{2\left(y_i - \ln(2)\right) + \ln(2),\ q_{0.0001}\left(\mathcal{E}(1)\right)\right\}.$$

   $\hookrightarrow$ Die Hazardtransformation verändert sich nicht, lediglich die Reihenfolge bzgl. $\leq_{\text{acc}}$.
2. **Quantil-basierte Kontamination:**
   - Kontaminiere Daten durch Ersetzen mit atypisch kleinen oder großen Werten bzgl. $\mathcal{E}(1)$.
   - Für eine Kontamination der *i*ten Beobachtung, ersetze y_i zufällig durch

   $$\tilde{y}_i = q_{0.0001}\left(\mathcal{E}(1)\right) \quad \text{oder} \quad \tilde{y}_i = q_{0.9999}\left(\mathcal{E}(1)\right).$$

- Im Beispiel aus dem Disputationsvortrag: 40% Quantil-basierte Kontamination.