On the Design and Performance of a Control Chart for Monitoring Continuous data in (0,1) when Process Parameters are Unknown

A.C. Rakitzis

University of Piraeus, Department of Statistics and Insurance Science, Piraeus, Greece
(Joint work with S. Chakraborti & N. Kumar)

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Scope & Outline

- Introduction
- The two-sided SH_B-chart
  - Case of known parameters (Case K)
  - Case of estimated parameters (Case U)
- Numerical Study
  - Performance in Case U of the SH_B-chart
- Adjustments
- Out-of-control performance
- Conclusions

- Control chart is the most efficient procedure in statistical process monitoring/control (SPM/C), suitable for deciding
  - if a process operates with only common causes of variation (that is, the process is stable & IC, whereas products meet the specifications) or
  - if it operates in the presence of assignable causes (that is, the process is unstable & OOC, with an excessive number of non-conforming products).
- **Implementation:** For $t = 1, 2, \ldots$, collect samples, $X_t = (X_{t1}, \ldots, X_{tn})$, $n \geq 1$, determine the value of a $g(\mathbf{X}_t)$ (e.g., $g(\mathbf{X}_t) = \overline{X}_t$ or X_t), plot $g(\mathbf{x}_t)$ on the chart, against one or more control limits, and declare the process as OOC according a rule.

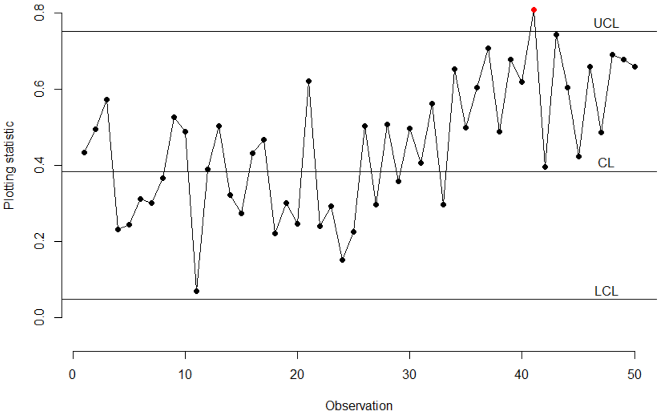

Figure: A two-sided control chart

- X: *critical-to-quality* characteristic (r.v.)
- There are two main categories of control charts: The variables charts (X is a continuous r.v.) and the attributes charts (X is a discrete r.v.)
- The usual assumption is that $X \sim \mathcal{N}(\mu, \sigma^2)$, but there are cases where the support of X is the $(0, \infty)$ (e.g. case of lifetime data) or a bounded interval, such as the $(0, 1)$.
- There are several processes where $X \in (0, 1)$:
  - relative humidity
  - proportion of a specific ingredient in a food or a pharmaceutical product
  - proportion of a raw material converted to a specific product during a chemical process.

- A term for data of this type is *continuous proportions* and there are several models that can be used for describing them, such as:
  - Beta distribution (Gupta and Nadarajah 2004)
  - Simplex distribution (Barndorff-Nielsen and Jørgensen 1991)
  - Unit-Gamma distribution (Grassia 1979)
  - Unit-Weibull distribution (Mazucheli et al. 2020)
  - Kumaraswamy distribution (Kumaraswamy 1980)
- Control charts for continuous proportions have been studied by several authors (e.g., Sant' Anna and ten Caten (2012), Bayer et al. (2018), Ho et al. (2019), Chowdhury et al. (2020), Lima-Filho et al. (2020), Knoth (2021), Lima-Filho and Bayer (2021), Rakitzis and Lafatzi (2025)) but mainly only in the case of *known parameters* (Case U).
- In this work, the focus is on **Beta distribution** as the appropriate model for data in $(0, 1)$.

- Usually the performance of a control chart is studied under the assumption that process parameters are known (**Case K**). However, in practice, this assumption is *rarely met* (the process parameters are unknown, **Case U**) and thus, they must be estimated from a preliminary sample (known also as Phase I sample).
- It is well known (Montgomery, 2020) that the theoretical performance of a control chart (i.e. in Case K) *is very different* than the performance in the case of estimated parameters (i.e. in Case U).
- **Aim:** Investigate how much is affected the theoretical performance of the two-sided Shewhart-type control chart for monitoring continuous proportions, following the Beta distribution, in the case of the estimated parameters.
  - How can we mitigate this effect?
  - What practitioners should take into account when designing their charts?

- Let $X \sim Beta(\theta_1, \theta_2)$. Then, for $x \in (0,1)$, its pdf is

$$f_{Beta}(x\,|\theta_1, \theta_2\,) = \frac{x^{\theta_1 - 1}(1-x)^{\theta_2 - 1}}{B(\theta_1, \theta_2)}$$

  where $\theta_1, \theta_2 > 0$ and $B(\theta_1, \theta_2)$ is the Beta function.
- Mean and Variance:

$$\mathbb{E}(X) = \frac{\theta_1}{\theta_1 + \theta_2}, \quad \mathbb{V}(X) = \frac{\theta_1 \theta_2}{(\theta_1 + \theta_2)^2(\theta_1 + \theta_2 + 1)}$$

- cdf of the Beta distribution:

$$F_{Beta}(x\,|\theta_1, \theta_2\,) = P(X \leq x) = \frac{B(x; \theta_1, \theta_2)}{B(\theta_1, \theta_2)}$$

  where $B(x; \theta_1, \theta_2) = \int_0^x t^{\theta_1 - 1}(1-t)^{\theta_2 - 1} dt$ is the incomplete Beta function.

The Two-Sided SH_B-Chart

Case K - The Beta (and re-parameterized Beta) distribution

- Re-parameterized Beta distribution (Rigby et al. (2019), pp. 461-463): Let $x \in (0,1)$ and set

$$\mu = \theta_1/(\theta_1+\theta_2), \;\; \sigma = 1/\sqrt{\theta_1+\theta_2+1}.$$

Then, if $X \sim RBeta(\mu,\sigma)$, its pmf is

$$f_{RBeta}(x\,|\mu,\sigma) = f_{Beta}\left(x\,\middle|\,\mu\left(\frac{1}{\sigma^2}-1\right),(1-\mu)\left(\frac{1}{\sigma^2}-1\right)\right)$$

where $\mu,\sigma \in (0,1)$.

- Also

$$\mathbb{E}(X)=\mu, \;\; \mathbb{V}(X)=\sigma^2\mu(1-\mu)$$

- μ : expected value, σ : precision parameter

The Two-Sided SH_B-Chart

Case K

- Let $\alpha \in (0,1)$ be the false alarm rate (i.e., probability of signal, given that the process is IC, or Type-I error).
- For an IC process: $\boldsymbol{\theta}_0 = (\mu, \sigma) = (\mu_0, \sigma_0)$. Also, $n = 1$ (individual observations).
  - In Case K we assume that μ_0 and σ_0 are known.
- The standard way to determine the control limits of the chart is to use the equal tail probabilities model, i.e.

$$P(X > UCL\,|\boldsymbol{\theta}_0) = P(X < LCL\,|\boldsymbol{\theta}_0) = \frac{\alpha}{2} \Rightarrow$$

$$LCL = F^{-1}_{RBeta}(\alpha/2; \mu_0, \sigma_0), \;\; UCL = F^{-1}_{RBeta}(1 - \alpha/2; \mu_0, \sigma_0)$$

- OOC signal at point t: if $X_t \notin [LCL, UCL]$.
- For an OOC process: $\mu_1 \neq \mu_0$ or $\sigma_1 \neq \sigma_0$.

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- β: probability of not signal, given that the process is OOC (Type-II error) equals

$$\begin{aligned}\beta &= P(X \in [LCL, UCL] \,|OOC) \\ &= F_{RBeta}(UCL \,|\mu_1, \sigma_1) - F_{RBeta}(LCL \,|\mu_1, \sigma_1)\end{aligned}$$

- For a Shewhart-type chart, in Case K, $RL \sim Ge(1-\beta)$, with pmf and cdf given by (for $\ell = 1, 2, \ldots$)

$$P(RL = \ell) = f_{RL}(\ell) = (1-\beta)\beta^{\ell-1},$$

$$P(RL \leqslant \ell) = 1 - \beta^{\ell}, \;\; \ell \geqslant 1$$

- The most common measure that is used to evaluate the performance of the control chart is the average run length, $\mathbb{E}(RL) = \mathrm{ARL}$.
- The ARL is defined as the expected number of points plotted on the chart until it gives an OOC signal for the first time.
- Other measures: Standard Deviation of the Run Length (SDRL).
- For the SH_B-chart, in Case K, it is

$$\mathrm{ARL} = \frac{1}{1-\beta}\,,\ \mathrm{SDRL} = \frac{\sqrt{\beta}}{1-\beta}$$

whereas, when the process is IC, it is

$$\mathrm{ARL} = \frac{1}{\alpha}\,,\ \mathrm{SDRL} = \frac{\sqrt{1-\alpha}}{\alpha}$$

- In practice, the process parameters μ_0 and σ_0 are unknown and must be estimated from a Phase I dataset.
- So far in the literature, there are no studies regarding the performance of a chart for continuous proportions in Case U.
  - However, there are many works and methods on how to evaluate the chart's performance in Case U, either for control charts for variables or for attributes (e.g., Zhao and Driscoll (2016), Does et al. (2020), Mosquera et al. (2021), Jardim et al. (2020), Sarmiento et al. (2022), Kumar (2022), Madrid-Alvarez et al. (2024), Huang (2022), Saghir et al. (2021), Mamzeridou and Rakitzis (2024) and references therein).

The Two-Sided SH_B-Chart

Case U - Maximum Likelihood Estimation

- Let $X_1, X_2, \ldots, X_m$ a Phase I dataset of size m from an IC process, where $X_i \sim RBeta(\mu_0, \sigma_0)$, $i = 1, 2, \ldots, m$.
- We assume that both μ_0, σ_0 are unknown and they are estimated via the Maximum Likelihood (ML) estimation method.
- The MLEs of μ_0, σ_0, say $\hat{\mu}$, $\widehat{\sigma}$, are obtained numerically (no-closed formulas). In `R` this can be implemented with the `gamlss` package.

The Two-Sided SH_B-Chart

Case U

- In Case U, the LCL, UCL depend on the estimates of μ_0, σ_0, as well as on the FAR α. We denote them as $\widehat{LCL}$, $\widehat{UCL}$.
- Let also $\boldsymbol{\theta}_0 = (\mu_0, \sigma_0)$ and $\hat{\boldsymbol{\theta}} = (\hat{\mu}, \widehat{\sigma})$. Then, given $\hat{\boldsymbol{\theta}}$ and α, the $\widehat{LCL}$, $\widehat{UCL}$ (or *plug-in* control limits) are calculated as

$$\widehat{LCL} = F^{-1}_{RBeta}(\alpha/2; \hat{\mu}, \widehat{\sigma}), \ \ \widehat{UCL} = F^{-1}_{RBeta}(1 - \alpha/2; \hat{\mu}, \widehat{\sigma})$$

- When process parameters are estimated, different Phase I samples, from the same IC process, will end up in different values for the $\widehat{LCL}$, $\widehat{UCL}$ (*practitioner-to-practitioner variability, P2P*) and different IC ARL values, conditional to the estimates $\hat{\mu}$, $\widehat{\sigma}$.
- To evaluate the performance of the SH_B-Chart in Case U, we need the distribution of the IC ARL (here, we use simulation to derive it).

The Two-Sided SH_B-Chart

Case U - Distribution of the IC CARL

1. Choose a Phase I sample of size m from a $RBeta(\mu_0, \sigma_0)$ process. Both μ_0, σ_0 are unknown.
2. Use the Phase I sample, estimate μ_0, σ_0 (MLE method) and let $\widehat{\mu}$, $\widehat{\sigma}$ be the estimates.
3. Given α, $\widehat{\mu}$ and $\widehat{\sigma}$, determine the *plug-in* control limits.
4. Calculate the in-control *conditional* ARL as

$$\mathrm{CARL}_0 = \frac{1}{1 - F_{RBeta}\left(\widehat{UCL}; \mu_0, \sigma_0\right) + F_{RBeta}\left(\widehat{LCL}; \mu_0, \sigma_0\right)}$$

5. Repeat Steps 1-4 N times (e.g. $N = 25000$) and estimate:

- the $\mathbb{E}(\mathrm{CARL}_0)$ as the average of the N values CARL_0.
- the $\sqrt{\mathbb{V}(\mathrm{CARL}_0)}$, as the stdev of the N values CARL_0.

Numerical Results

- $m \in \{30, 50, 100, 200, 500, 1000, 2000, 5000\}$.
- We assume $\alpha = 0.0027$ (nominal FAR) and the theoretical IC ARL (in Case K) is 370.4.
- 4 different IC scenarios:
  $(\mu_0, \sigma_0) \in \{(0.15, 0.2), (0.35, 0.3), (0.5, 0.08), (0.75, 0.4)\}$
- Performance measures:
  - $\mathrm{AARL}_0 = \frac{1}{N}\sum_{l=1}^{N} \mathrm{CARL}_{0,l}$.
  - q-percentile points, $q \in \{0.05, 0.25, 0.5, 0.75, 0.95\}$ of the distribution of CARL_0.
  - percentage of cases $\{\mathrm{CARL}_0 < 370.4\}$.

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Numerical Results

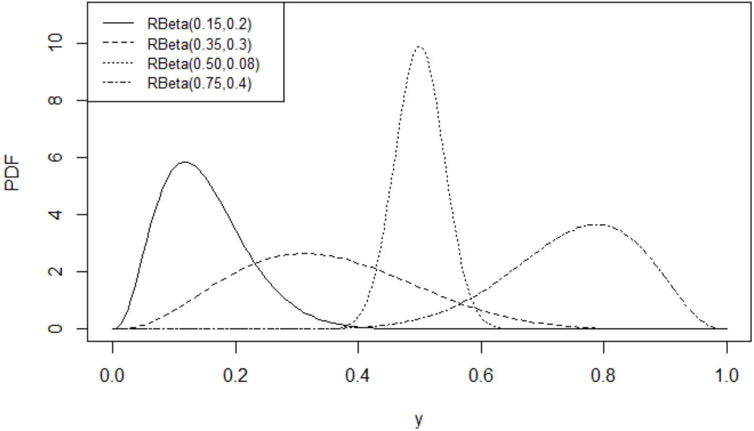
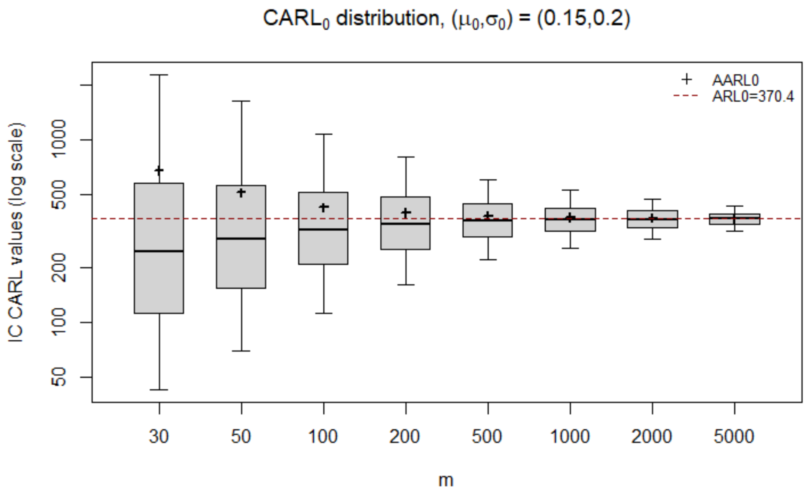


Figure: The 4 different IC scenarios

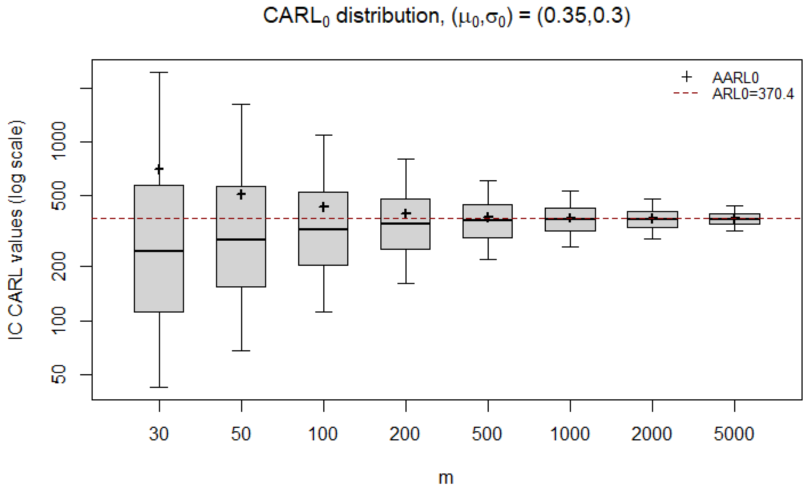
Conditional IC ARL distribution

MLE method, $\alpha = 0.0027$, Scenario 1, *plug-in* limits



Conditional IC ARL distribution

MLE method, $\alpha = 0.0027$, Scenario 2, *plug-in* limits



Adjusted & Corrected Limits

- **Main (practical) results:**
  - As $m \nearrow$ the variability of the distribution of $\mathrm{CARL}_0 \searrow$.
  - The size m of the Phase I sample needs to be large enough (e.g. > 1000 or even > 5000) in order to have small differences in the IC performance of the two-sided SH_B-chart between Case K and Case U.
- In practice, it is not always possible to have such large sample sizes.
- For all the examined sizes m of the Phase I sample

$$\widehat{P}(CARL_0 < 370.4) \geq 0.5$$

- Next, we provide adjusted (or "corrected") values α' and α'' to be used instead of α, when m is pre-specified, using two different approaches.

- The following steps are suggested:
  1. Choose a Phase I sample of size m from a $RBeta(\mu_0, \sigma_0)$ process with unknown parameters.
  2. Use the Phase I sample and estimate μ_0, σ_0. Let $\widehat{\mu}$, $\widehat{\sigma}$ the ML estimates.
  3. Given m, $\widehat{\mu}$, $\widehat{\sigma}$, find α' such that

$$RD = \frac{AARL_0 - ARL_0}{ARL_0} < p', \;\; p' \in (0,1)$$

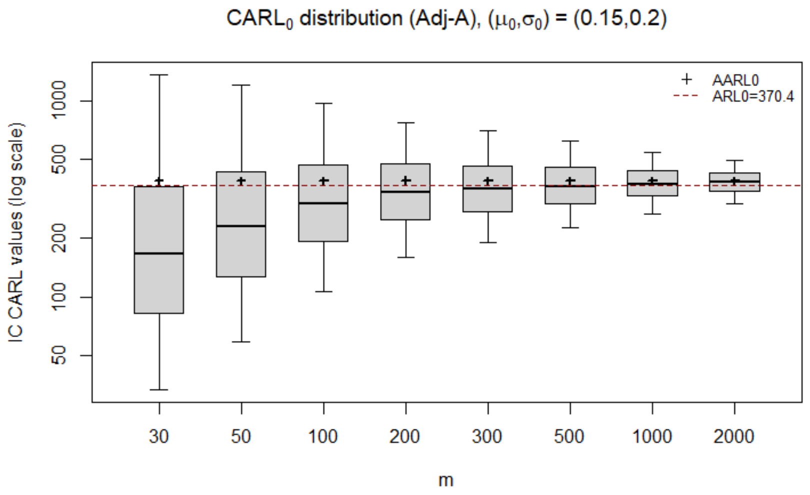
We will refer to this as **Adj-A**. The limits are calculated as

$$\widehat{LCL} = F^{-1}_{RBeta}(\alpha'/2; \widehat{\mu}, \widehat{\sigma}), \;\; \widehat{UCL} = F^{-1}_{RBeta}(1-\alpha'/2; \widehat{\mu}, \widehat{\sigma})$$

- ⋆ The steps are the same as described before (for the calculation of $AARL_0$), for several values $\alpha' \neq \alpha$.
- ⋆ For illustrative purposes, we use as $ARL_0 = 370.4$ and $p' = 0.05$ (i.e. a 5% difference).

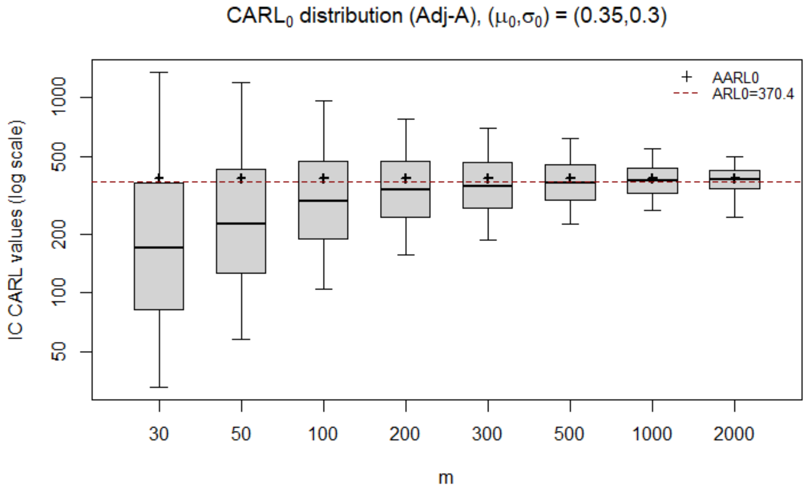
Conditional IC ARL distribution

Adj-A, Scenario 1, $p' = 0.05$



Conditional IC ARL distribution

Adj-A, Scenario 2, $p' = 0.05$



Adjustment B

Statistical Design based on an adjusted FAR α''

- The following steps are suggested:
  1. Choose a Phase I sample of size m from a $RBeta(\mu_0, \sigma_0)$ process with unknown parameters.
  2. Use the Phase I sample and estimate μ_0, σ_0. Let $\widehat{\mu}$, $\widehat{\sigma}$ the ML estimates.
  3. Given m, $\widehat{\mu}$, $\widehat{\sigma}$, find α'' such that (see also Sarmiento et al. (2022))

$$\frac{\#\{CARL_0 < (1+\epsilon)^{-1} ARL_0\}}{N} < p'$$

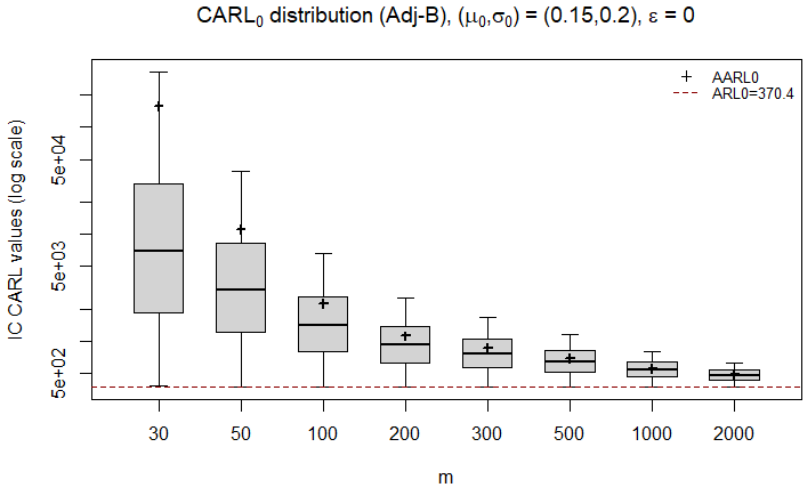
  We will refer to this as **Adj-B**.

⋆ The steps are the same as described previously and they are applied for several values $\alpha'' \neq \alpha$ of the FAR.

⋆ For illustrative purposes, we consider $(\epsilon, p') = (0, 0.05) \rightsquigarrow$ we want only 5% of charts to have IC $\text{ARL} < 370.4$.

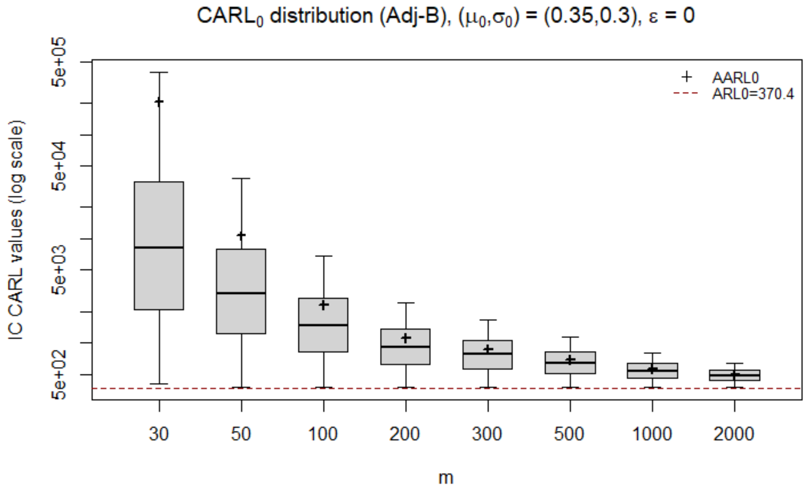
Conditional IC ARL distribution

Adj-B, Scenario 1, $(\epsilon, p') = (0, 0.05)$



Conditional IC ARL distribution

Adj-B, Scenario 2, $(\epsilon, p') = (0, 0.05)$



OOC performance

$AARL_1$ ($SDARL_1$)

- Multiplicative shifts in μ_0, σ_0, $\mu_1 = \tau_1 \mu_0$, $\sigma_1 = \tau_2 \sigma_0$, with $\tau_1, \tau_2 > 0$ such that $\mu_1, \sigma_1 \in (0,1)$.
- In Case K:

$$\mathrm{ARL}_1 = \frac{1}{1 - F_{RBeta}(UCL; \mu_1, \sigma_1) + F_{RBeta}(LCL; \mu_1, \sigma_1)}$$

- In Case U: OOC conditional ARL

$$\mathrm{CARL}_1 = \frac{1}{1 - F_{RBeta}(\widehat{UCL}; \mu_1, \sigma_1) + F_{RBeta}(\widehat{LCL}; \mu_1, \sigma_1)}$$

- Performance measures in Case U

$$\mathrm{AARL}_1 = \frac{1}{N} \sum_{l=1}^{N} \mathrm{CARL}_{1,l},$$

$$\mathrm{SDARL}_1 = \sqrt{\frac{1}{N-1} \sum_{l=1}^{N} (\mathrm{CARL}_{1,l} - \mathrm{AARL}_1)^2}$$

OOC performance

$AARL_1$ ($SDARL_1$), $m = 100$, $(\mu_0, \sigma_0) = (0.15, 0.2)$

τ_1	τ_2	ARL_1 $\alpha = 0.0027$	*plug-in* $\alpha = 0.0027$	**Adj-A** $\alpha' = 0.00297$	**Adj-B**, $\epsilon = 0$ $\alpha'' = 0.00059$
0.9	0.8	3761.47	5855.26 (9498.18)	5056.85 (9771.78)	77706.83 ($> 10^5$)
0.9	0.9	779.92	1003.47 (1097.47)	891.93 (1039.83)	7070.99 (13622.14)
0.9	1.0	247.77	287.94 (232.25)	261.77 (215.86)	1335.68 (1671.98)
0.9	1.1	104.35	114.46 (71.93)	105.84 (66.84)	395.84 (359.98)
0.9	1.2	53.36	56.55 (28.70)	52.97 (26.81)	157.63 (111.12)
1.0	0.8	6619.44	10352.38 (18826.55)	8738.39 (18416.66)	$> 10^5$ ($> 10^5$)
1.0	0.9	1254.06	1612.82 (1926.69)	1412.06 (1775.32)	13151.87 (27102.50)
1.0	1.0	370.37	430.38 (374.75)	387.00 (341.51)	2258.87 (3027.91)
1.0	1.1	147.11	161.50 (109.04)	148.09 (99.80)	620.15 (603.94)
1.0	1.2	71.76	76.16 (41.47)	70.87 (38.26)	232.08 (175.31)
1.1	0.8	4194.12	7552.52 (14691.47)	6378.77 (12841.37)	$> 10^5$ ($> 10^5$)
1.1	0.9	996.80	1381.48 (1741.18)	1210.02 (1537.22)	10921.05 (23354.68)
1.1	1.0	339.47	409.69 (372.92)	368.21 (333.31)	2127.98 (2914.37)
1.1	1.1	146.99	164.43 (115.39)	150.59 (104.42)	634.39 (630.52)
1.1	1.2	75.41	80.79 (45.60)	75.05 (41.73)	249.94 (193.15)

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OOC performance

$AARL_1$ ($SDARL_1$), $m = 200$, $(\mu_0, \sigma_0) = (0.35, 0.3)$

τ_1	τ_2	ARL_1 $\alpha = 0.0027$	*plug-in* $\alpha = 0.0027$	**Adj-A** $\alpha' = 0.00276$	**Adj-B**, $\epsilon = 0$ $\alpha'' = 0.00101$
0.9	0.8	4069.77	5183.07 (5057.68)	4975.45 (4797.98)	24684.12 (34143.33)
0.9	0.9	807.92	921.56 (636.59)	893.44 (611.63)	3060.53 (2730.56)
0.9	1.0	246.42	265.41 (138.79)	259.04 (134.43)	686.99 (446.89)
0.9	1.1	99.96	104.33 (42.98)	102.31 (41.86)	225.39 (113.32)
0.9	1.2	49.40	50.64 (16.88)	49.84 (16.50)	95.50 (38.40)
1.0	0.8	8403.62	10315.81 (10256.37)	9936.45 (9986.50)	56368.27 (77922.18)
1.0	0.9	1379.97	1547.04 (1103.66)	1501.83 (1074.65)	5699.08 (5275.47)
1.0	1.0	370.37	395.79 (215.18)	386.47 (210.00)	1109.49 (755.63)
1.0	1.1	137.45	142.98 (61.51)	140.22 (60.18)	328.75 (173.63)
1.0	1.2	63.73	65.25 (22.78)	64.20 (22.34)	129.28 (54.71)
1.1	0.8	4434.98	5869.03 (6083.25)	5687.01 (6161.88)	29227.22 (43357.07)
1.1	0.9	940.61	1094.79 (800.55)	1067.19 (790.15)	3786.78 (3617.99)
1.1	1.0	296.69	322.81 (178.06)	316.11 (174.98)	868.73 (600.75)
1.1	1.1	121.83	127.80 (55.45)	125.58 (54.50)	286.63 (152.58)
1.1	1.2	60.17	61.83 (21.72)	60.93 (21.37)	120.83 (51.39)

OOC performance

$AARL_1$ $(SDARL_1)$, $m = 1000$, $(\mu_0, \sigma_0) = (0.35, 0.3)$

τ_1	τ_2	ARL_1 $\alpha = 0.0027$	*plug-in* $\alpha = 0.0027$	**Adj-A** $\alpha' = 0.00261$	**Adj-B**, $\epsilon = 0$ $\alpha'' = 0.00182$
0.9	0.8	4069.77	4279.95 (1543.99)	4492.16 (1605.00)	7614.99 (2932.70)
0.9	0.9	807.92	830.98 (229.01)	863.32 (236.02)	1305.67 (383.85)
0.9	1.0	246.42	250.49 (54.43)	258.32 (55.73)	360.09 (83.38)
0.9	1.1	99.96	100.94 (17.70)	103.53 (18.04)	135.83 (25.36)
0.9	1.2	49.40	50.75 (7.27)	49.70 (7.16)	63.57 (9.74)
1.0	0.8	8403.62	8761.33 (3247.59)	9233.13 (3384.20)	16482.22 (6508.95)
1.0	0.9	1379.97	1413.67 (402.74)	1473.28 (415.74)	2316.22 (701.94)
1.0	1.0	370.37	375.82 (84.88)	388.51 (87.02)	557.81 (133.89)
1.0	1.1	137.45	138.71 (25.39)	142.52 (25.90)	191.34 (37.20)
1.0	1.2	63.73	64.09 (9.67)	65.56 (9.83)	83.62 (13.39)
1.1	0.8	4434.98	4694.83 (1782.79)	4930.15 (1863.32)	8481.28 (3438.76)
1.1	0.9	940.61	970.80 (281.59)	1009.30 (291.19)	1548.73 (478.06)
1.1	1.0	296.69	302.09 (69.03)	311.80 (70.85)	440.94 (107.01)
1.1	1.1	121.83	123.13 (22.71)	126.40 (23.18)	168.18 (32.90)
1.1	1.2	60.17	60.55 (9.19)	61.90 (9.34)	78.58 (12.63)

ΠΑΝΕΠΙΣΤΗΜΙΟ ΠΕΙΡΑΙΩΣ
ΤΜΗΜΑ ΣΤΑΤΙΣΤΙΚΗΣ & ΑΣΦΑΛΙΣΤΙΚΗΣ ΕΠΙΣΤΗΜΗΣ
ΕΡΓΑΣΤΗΡΙΟ ΣΤΑΤΙΣΤΙΚΗΣ

Conclusions I

- We studied the performance of a two-sided chart for individual observations from Beta distribution, when both process parameters are unknown and must be estimated.
- To assess the difference between the Case K and the Case U we used the conditional IC ARL distribution.
- The results showed that Phase I samples with $m \leq 50$ are not suggested due to the increased variability in the distribution of the IC CARL.
- For $m \geq 500$ the difference between $\mathbb{E}(\mathrm{CARL}_0)$ and ARL_0 is $< 5\%$ whereas only for $m \geq 5000$ the

$$\sqrt{\mathbb{V}(\mathrm{CARL}_0)} \approx 10\% \times \mathrm{ARL}_0.$$

- Also, when *plug-in limits* are used, the % of cases $\{\mathrm{CARL}_0 < 370.4\}$ exceeds 50%, even for very large Phase I samples.

Conclusions II

- When m is pre-specified (and *not very large*), we suggest limits' adjustment:
  - the $\mathbb{E}(\mathrm{CARL}_0)$ is close to the nominal ARL_0 (**Adj-A**).
  - the % of cases $\{\mathrm{CARL}_0 < 370.4\}$ is small (**Adj-B**).
- We investigated how the OOC performance is affected when using different adjusted control limits.
  - Using wider control limits (Adj-B case) guarantees the IC performance of the chart (less false alarms) but affects its ability in detecting shifts in process parameters
  - Under Adj-A the % of cases $\{\mathrm{CARL}_0 < 370.4\}$ can be close to 50% (increased number of false alarms) but the OOC performance, in terms of OOC $\mathbb{E}(\mathrm{CARL})$ is close to that under Case K.
  - It is up to the practitioners to decide if and how to adjust the limits.

Thank you very much for your attention!

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